



Interest Rates

Chapter 4

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Outline

- Types of interest rates
- Compounding frequency
- Spot rates
- How to obtain spot rates
- Forward rates and FRAs
- Theory of the term structure of interest rates (yield curve)

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Types of Rates

- US Treasury rates: rates an investor earns investing in Treasury bills, notes, and bonds
- LIBOR rates: London interbank offer rate
- LIBOR quoted by a particular bank is the rate of interest at which the bank is prepared to make a large wholesale deposit with other banks
- AA-rated financial institutions regard LIBOR as their short-term opportunity cost of capital

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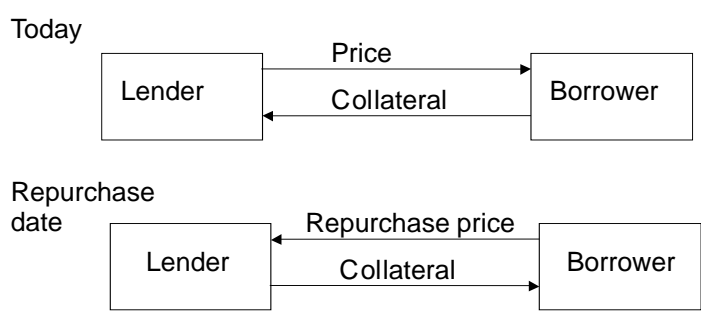


Types of Rates

- LIBOR is the lending rate between large banks in the Eurodollar market
- Eurodollar market is the market that trades US dollar outside of the United states
- LIBID: London interbank bid rate
- LIBID is the rate a bank is prepared to accept deposits from other banks
- Money demand and supply sets these rates

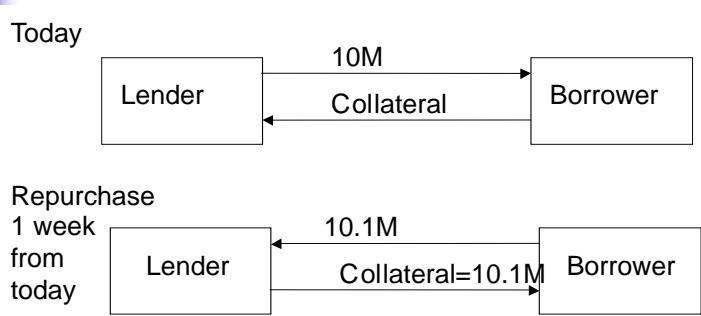
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Repurchase agreement (Repo)



- From borrower's side: Repo; From lender's side: Reverse repo

Repurchase agreement (Repo)



- From borrower's side: Repo; From lender's side: Reverse repo



Risk to lender using Repo

- Collateral changes in price
- Lend principal at or lower than the price of collateral by the amount determined as margin
- Set: repurchase price = price of collateral – margin
- Credit risk of lender is less important than lending without collateral

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Measuring Interest Rates

- The compounding frequency used for an interest rate is the unit of measurement
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers

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Measuring Interest Rates

- Bank I offers 10% interest rate compounded annually
- Bank II offers 10% interest rate compounded daily
- Investment of \$1000 today in 1 year will grow to:
 - Bank I: after 1 year $\$1000(1+0.1) = 1100$
 - Bank II: \$1000 → after 1 day $1000(1+0.1/365)$ → after 2 days $1000(1+0.1/365)^2$ → after 1 year $\$1000(1+0.1/365)^{365} = 1105.2$

As an investor, which do you prefer?

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Measuring Interest Rates

- Bank I must offer 10.52% compounded annually to be competitive and attract investors.
- In competitive markets, interest rates of different compounding period must offer the same investment growth

$$(1+r_{\text{yearly}})^T = \left(1 + \frac{r_{\text{daily}}}{365}\right)^{T \cdot 365} = \left(1 + \frac{r_{\text{semi-annual}}}{2}\right)^{T \cdot 2}$$

- It is customary to quote rates multiplied by its compounding period (annualized).

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Concept check



- You want to invest your money for one year. Which of the following interest rates do you prefer?

G) 4% annual percentage rate
Y) 3.98% semi-annual rate
R) 3.85% monthly rate

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Measuring Interest Rates

- Where do we get the competitive interest rates (discount rates)?
- The present value of \$1 at time T in the future is set by the market when bonds are traded
- We can obtain the present value of \$1 from zero coupon bonds or from coupon bonds traded in the market

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Measuring Interest Rates

Mat.	Price	T	d(T) *
0.5	97.087	0.5	0.97087
1	93.897	1	0.93897
1.5	90.909	1.5	0.90909
2	87.336	2	0.87336

* $d(T) = \text{Price of zero-coupon bonds} / \text{Face Value}$

Define the discount $d(T)$ as the present value of \$1 received at time T.

The discount factor can be used to quote spot rates of any compounding frequency

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Compounding frequency

- The discount factor $d(2) = 0.87336$ is equivalent to the following compounding rates

t=0	0.5	1.0	1.5	2.0
0.87336				1.0
0.87336	0.935			1.0
0.87336	0.904	0.935	0.967	1.0

- Rates of higher compounding intervals are lower because the interest payments get reinvested more often

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Conversion Formulas

- If the rate is compounded m times per annum, the terminal value of the investment is

$$\left(1 + \frac{R}{m}\right)^{nm}$$

- By equating the terminal values we can convert one compounding rate to another

$$(1 + r_{\text{yearly}})^T = \left(1 + \frac{r_{\text{daily}}}{365}\right)^{T \cdot 365} = \left(1 + \frac{r_{\text{semi-annual}}}{2}\right)^{T \cdot 2}$$

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Conversion formulas

R_c : continuously compounded rate

R_m : same rate with compounding m times per year

$$e^{nR_c} = \left(1 + \frac{R_m}{m}\right)^{mn}$$

$$R_c = m \ln \left(1 + \frac{R_m}{m}\right)$$

$$R_m = m \left(e^{R_c/m} - 1\right)$$

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Continuous compounding

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates
- \$100 grows to $\$100e^{RT}$ when invested at a continuously compounded rate R for time T
- \$100 received at time T discounts to $\$100e^{-RT}$ at time zero when the continuously compounded discount rate is R

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Obtaining spot rates from coupon bonds

Bond Principal (dollars)	Time to Maturity (years)	Annual Coupon (dollars)	Bond Price (dollars)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

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The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.
- The 3-month rate is 4 times 2.5/97.5 or 10.256% with quarterly compounding
- This is 10.127% with continuous compounding
- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding

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The Bootstrap Method

- To calculate the 1.5 year rate we solve

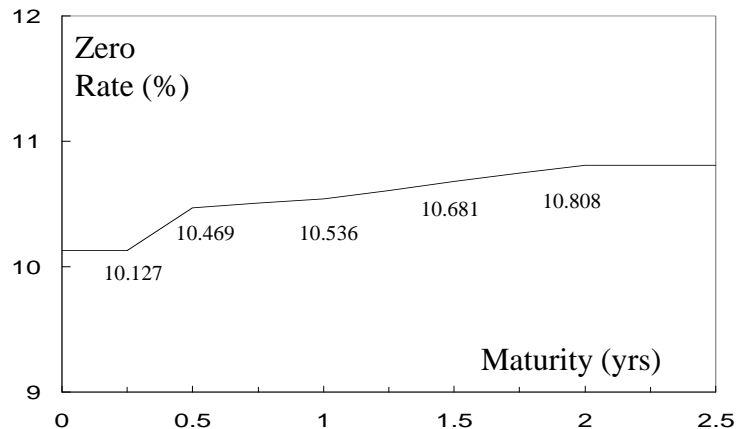
$$4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96$$

to get $R = 0.10681$ or 10.681%

- Similarly the two-year rate is 10.808%
- For a comprehensive reading on how bond traders derive the spot rates see Chapter 4 in Fixed Income Securities by Bruce Tuckman

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Zero Curve Calculated from the Data (Term structure of interest rates)



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Zero rates or spot rates

- A zero rate (or spot rate), for maturity T is the rate of interest earned on an investment that provides a payoff only at time T
- It is the rate an investor gets by buying and holding a zero-coupon bond to maturity
- Spot rate at T is also used to obtain the present value of cash flow at each time T
- Spot rates can be quoted in any compounding frequency

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The yield curve

Deriving your own yield curve vs. using published yield curves

- Published yield curve
 - Often too coarse
 - Easily obtained
- Deriving your own yield curve
 - Higher accuracy
 - A more complex process in practice because there are many traded bonds

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Bond Pricing



- You have a project with the US government. You will receive \$100,000 from this project in 2 years. What is the present value of this cash flow?

Maturity	yield (annual comp.)
1	5%
2	6%

G) $100K/(1.05)(1.06)$

Y) $100K/(1.06)^2$

R) $100K/(1.06)$

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Example (Table 4.2, page 81)

Maturity (years)	Zero Rate (% cont comp)
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

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Bond Pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate
- In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is

$$3e^{-0.05 \times 0.5} + 3e^{-0.058 \times 1.0} + 3e^{-0.064 \times 1.5} \\ + 103e^{-0.068 \times 2.0} = 98.39$$

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Bond Yield (yield to maturity)

- Yield to maturity is defined as the single discount rate that makes the present value of the cash flows of a bond equal to the market price of this bond
- Suppose that the market price of the bond in our example equals its theoretical price of 98.39
- The bond yield is given by solving

to get $y=0.0676$ or 6.76%.

$$3e^{-y \times 0.5} + 3e^{-y \times 1.0} + 3e^{-y \times 1.5} + 103e^{-y \times 2.0} = 98.39$$

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Bond Yield (yield to maturity)

- Bond quote in the WSJ

GOVT. BOND & NOTES						
RATE	MATURITY	BID	ASKED	CHG.	ASKED	F
	MO/YR				YLD.	
5 1/2	Jul 01n	100:00	100:02	1.68	
6 5/8	Jul 01n	100:00	100:02	- 1	2.76	
7 7/8	Aug 01n	100:07	100:00		2.92	

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Par Yield

- The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.
- In our example we solve

$$\frac{c}{2}e^{-0.05 \times 0.5} + \frac{c}{2}e^{-0.058 \times 1.0} + \frac{c}{2}e^{-0.064 \times 1.5} + \left(100 + \frac{c}{2}\right)e^{-0.068 \times 2.0} = 100$$

to get $c=6.87$ (with semi - annual compounding)

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Forward Rates

- The forward rate is the future zero rate implied by today's yield curve
- The forward rate is the interest rate in the mn^{th} period which equates the holding return of
 - (a) An mn -period zero-coupon bond and
 - (b) The strategy of holding an $(mn-1)$ period zero-coupon bond $mn-1$ periods and rolling it over through the mn^{th} period

$$(1 + f_{mn})(1 + \frac{R_{nm}}{m})^{mn-1} = (1 + \frac{R_{nm}}{m})^{mn}$$

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Forward Rates

- R_{mn} is the spot interest rate at period mn and f_{mn} is the forward rate for the period $mn-1$ through mn

- For annual rates $f_n = \frac{(1+R_n)^n}{(1+R_{n-1})^{n-1}} - 1$

- The continuous compounding rates

$$e^{(T_n - T_{n-1})f_n} e^{R_{n-1}T_{n-1}} = e^{R_n T_n}$$

- and

$$f_n = \frac{(R_n T_n - R_{n-1} T_{n-1})}{T_n - T_{n-1}}$$

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Calculation of Forward Rates

TIR:	T	Price	zero rates (annual)	forward rates
	1	934.58	7%	7%
	2	857.34	8%	9%
	3	782.91	8.5%	9.5%

Calculating forward rates $f_2 = \left[\frac{(1+.08)^2}{(1+.07)^1} - 1 \right] = 9\%$

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Calculation of Forward Rates

Table 4.5, page 84 (continuous rates)

Year (n)	Zero Rate for an n -year Investment (% per annum)	Forward Rate for n th Year (% per annum)
1	10.0	
2	10.5	11.0
3	10.8	11.4

$$f_n = \frac{(R_n T_n - R_{n-1} T_{n-1})}{T_n - T_{n-1}} = f_2 = \frac{(10.5 * 2 - 10 * 1)}{2 - 1} = 11$$

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Calculation of Forward Rates

Table 4.5, page 84 (continuous rates)

Year (n)	Zero Rate for an n -year Investment (% per annum)	Forward Rate for n th Year (% per annum)
1	10.0	
2	10.5	11.0
3	10.8	11.4
4	11.0	11.6
5	11.1	11.5

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Upward vs Downward Sloping Yield Curve

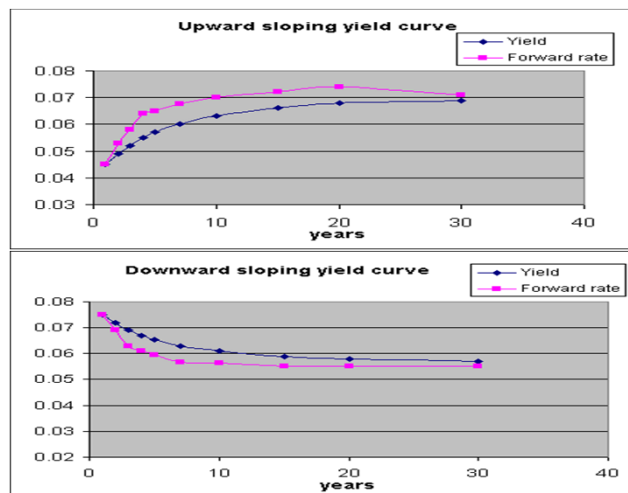
- The forward rate equation can also be written as

$$f_n = R_{n-1} + (R_n - R_{n-1}) \frac{(T_{n-1})}{T_n - T_{n-1}}$$

- For an upward sloping yield curve:
Forward Rate > Zero Rate
- For a downward sloping yield curve:
Forward rate > Zero Rate

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Upward vs Downward Sloping Yield Curve



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Forward Rates

- Forward rates equals the expected short rate in the future when interest rates are certain, but in general forward rates do not equal future short rates
- Why use forward rate?
 - Useful for understanding the theories of the TIR (yield curve)
 - Use in a forward rate agreement

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Forward Rate Agreement

- A forward rate agreement (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future time period

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Concept check



- Your company will work on a short project that starts one year from today and ends 2 years from today. It will need a loan at the start of the project and will get all the revenues to pay back the loan at the end of the project. The analysis of the project is based on current interest rates so you want to fix the future borrowing rate **today**. How would you use positions in the 1-year and 2-year zero-coupon bonds to fix your borrowing rate between year 1 and 2 (disregard your default risk for now)?

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Concept check



- G) Long 1-year zero-coupon Treasury bond and short 2-year zero-coupon Treasury bond
- Y) Short 1-year zero-coupon Treasury bond and long 2-year zero-coupon Treasury bond
- R) Long 1-year zero-coupon Treasury bond and long 2-year zero-coupon Treasury bond

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Forward rate agreement

You want to fix your borrowing rate today for borrowing \$1000 between years 1 and 2.0. Cash flow pattern?

Cash flow	↓	↑	
0	1000	?	
0	1	2.0	years

Long a 1-year zero coupon bond

Short a 2-year zero coupon bond

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Forward rate agreement

t	price	yield (annual)	forward (annual)
1	952.38	5%	5%
2	898.45	5.5%	6.0
3	839.62	6%	7.12

Cash flow (year)	0	1	2
Buy 1 one-year zero	-952.38	1000	
Short 1.06 (952.38/898.45) of 2.0 year zero	952.38	0	-1060
Net	0	1000	-1060
Interest rate = (1060-1000)/1000 =			6%

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Valuation of FRAs

- R_K : The rate of interest agreed to in the FRA
- R_F : The forward LIBOR interest rate for the period between T_1 and T_2 calculated today
- R_M : The LIBOR interest rate actually observed in the market at T_1 for the period T_1 to T_2
- L : The principal underlying the contract
- R_K , R_F , and R_M are measured with compounding frequency reflecting their maturity T_1 to T_2 .
- This means that if T_1 to T_2 is 6 months these are semiannual rates

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Valuation of FRAs

- A company X enters into a FRA to receive interest of R_K when the prevailing interest rate at T_1 is R_M , therefore the value of FRA to company X at T_1 is

$$L(R_K - R_M)(T_2 - T_1)$$

- The value of this FRA to the company (Y) who issues this FRA is

$$L(R_M - R_K)(T_2 - T_1)$$

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Valuation of FRAs

- In general FRAs are settle at time T_1 , therefore the value of the contract at closing is

$$\frac{L(R_K - R_M)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

- Suppose a company X enters into a FRA to **receive** fixed rate of 4% on a principal of 1M for a three-month period starting in three years
- If the 3-month LIBOR (quarterly comp.) at the end of three years is 4.5% the cash flow to X is

$$\frac{1M(0.04 - 0.045)(0.25)}{1 + 0.045(0.25)}$$

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Valuation of FRAs

- Compare two FRAs. The first promises that the LIBOR forward rate R_K will be earned for the period T_1 to T_2 . The second promises that the rate R_F will be earned. The two are exactly the same except for the interest earned at T_2 . So the value of the FRA is the present value of the difference in interest earned.

$$V_{FRA} = L(R_K - R_F)(T_2 - T_1)e^{-R_2T_2}$$

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Valuation of FRAs

- Suppose the spot rates and forward rates are

Year	spot(% cont.)	forward(%)
1	3	
2	4	5
3	4.6	5.8
4	5	6.2

Consider an FRA where we will receive a rate of 6% (annual comp.) on principal of 1M between end of year 1 and end of year 2

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Valuation of FRAs

- 5% continuous comp. equals 5.127 annual comp. We have

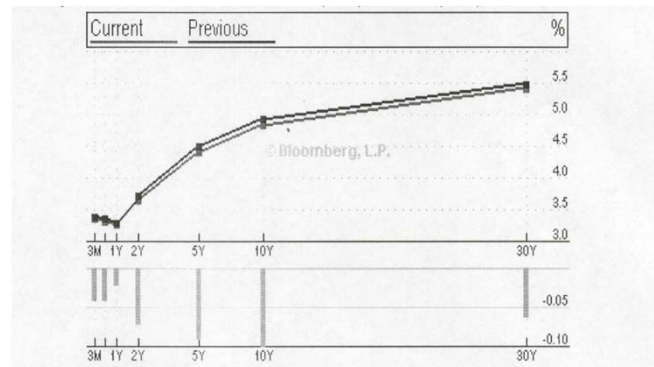
$$V_{FRA} = 1M(0.06 - 0.05127)(1)e^{-0.04*2} = 8,058$$

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Theories of the yield curve

What can we deduce from the shape of the yield curve?



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Theories of the yield curve

■ Expectations Theory:

- The yield curve is determined solely by expectations of future interest rates
- Forward rates equal expected future zero rates

■ Liquidity Preference Theory

- The yield curve is determined by expectations of future interest rates and different levels of risk associated with bonds of different maturities

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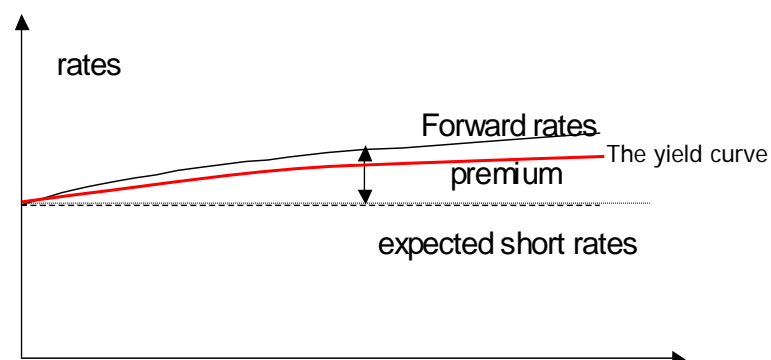
Liquidity preference theory

- A short-term investor incurs interest rate risk when he invests in long-term bonds
- A short-term investor will demand a premium for holding interest rate risk from investing in long-term bonds
- Most investors are long-term borrowers and shorter-term investors
- The forward rate equals the expected short rate plus a risk premium

$$f_n = E[\tilde{r}_n] + \text{premium}$$

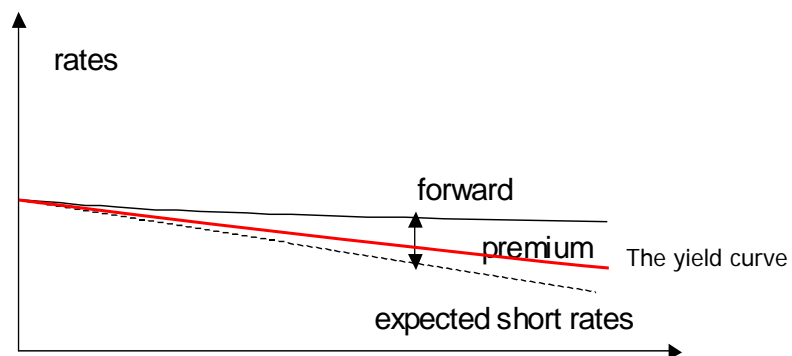
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Liquidity preference theory



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Liquidity preference theory



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Summary

- Types of interest rates
- Compounding frequency
- Spot rates
- How to obtain spot rates
- Forward rates and FRAs
- Theory of the term structure of interest rates (yield curve)

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