The Logic of Quantified Statements

1 Introduction & Definitions

Definition 1.1 (**Predicates & its variables).** .

- A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
- The **domain of a predicate variable** is the set of all values that may be substituted in place of the variable in the *predicate*.

Example 1.1. Let $Q(x, y) : x = y + 3$ with domain the collection of numbers $0, 1, 2, 3 \ldots$. What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Solution:

Example 1.2. Let $P(x)$ be the predicate "$x^3 > x$" with domain being the set of all real numbers, $\mathbb{R}$. Write

- $P(1)$ and $P(-1)$
- $P(\frac{1}{3})$
- $P(3)$

explicitly and indicate which of these statements are true and which are false.
What are the possible values of $x \in \mathbb{R}$ that make $P(x)$ true?

Solution:

Notations: $\mathbb{R}, \mathbb{Z}, \mathbb{N}$

- $\mathbb{R}$ is the set of all real numbers.
- $\mathbb{R}^+$ is the set of all positive real numbers.
- $\mathbb{R}^-$ is the set of all negative real numbers.
- $\mathbb{Z}$ is the set of all integers.
- $\mathbb{Z}^+$ is the set of all positive integers.
- $\mathbb{Z}^-$ is the set of all negative integers.
- $\mathbb{Q}$ is the set of all rational numbers. I.e. $\mathbb{Q} = \left\{\frac{a}{b} | a \in \mathbb{Z}, b \in \mathbb{Z} - \{0\}\right\}$
- $\mathbb{N}$ is the set of *natural* numbers. In general, $\mathbb{N}$ is just the set of all positive integers. However, it is possible to have 0 in some text books.

Definition 1.2.

- If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

  The **truth set** of $P(x)$ is denoted

  $$\{x \in D | P(x)\}$$

  which is read "the set of all x in D such that $P(x)$ is true."

Example 1.3. Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

1. the domain of n is the set $\mathbb{Z}^+$ of all positive integers,
2. the domain of n is the set $\mathbb{Z}$ of all integers,
3. the domain of n is the set $\{3, 5, 11, 16\}$.

Solution:

2 Universal Quantifier: $\forall$

Another way to generate propositions is by means of quantifiers. Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true

Definition 2.1. Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form

$$\forall x \in D, Q(x),$$

where the symbol $\forall$ denotes "for all" or "for any."

- It is defined to be **true** if, and only if, $Q(x)$ is true for **every** x in D.
- It is defined to be **false** if, and only if, $Q(x)$ is false for **at least one** x in D.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example 2.1. .

(i) Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, \quad x^2 \geq x.$$

Show that this statement is true.

(ii) Consider the statement

$$\forall x \in \mathbb{R}, \qquad x^2 \geq x.$$

Find a counterexample to show that this statement is false.
Solution:

3 The Existential Quantifier: ∃

Definition 3.1. Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form

$$\exists x \in D \quad \text{such that} \quad Q(x).$$

where the symbol ∃ denotes "there exists."

- It is defined to be **true** if, and only if, $Q(x)$ is **true** *for at least one* x in D.
- It is **false** if, and only if, $Q(x)$ is **false** for all x in D.

Example 3.1. Consider the statement

$$\exists x \in \mathbb{Z}^+ \quad \text{such that} \quad x^2 = x.$$

Show that this statement is true.
Solution:

Example 3.2. Let $D = \{5, 6, 7, 8\}$. Determine the truth value of the following statements:

$$\exists x \in D \quad \text{such that} \quad x^2 = x.$$

Solution:

Example 3.3. Let $P(x)$ denote the statement $x > 3$: Determine the truth value of the statement:

$$\exists x \in \mathbb{R}, P(x).$$

Solution:

Example 3.4. Determine the truth value of the following statements
1. $\forall x \in \mathbb{R},\ x > \frac{1}{x}$
2. $\exists x \in \mathbb{R},\ x > \frac{1}{x}$
3. $\forall x \in \{2, 3, 4\}, \quad x > \frac{1}{x}$
4. $\exists x \in \{-3, -2\}, \quad x > \frac{1}{x}$.

Solution:

4 Formal Versus Informal Language

It is important to be able to translate from formal to informal language when trying to make sense of mathematical concepts that are new to you. It is equally important to be able to translate from informal to formal language when thinking out a complicated problem.

Example 4.1. Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol "∀" or "∃."

(a) $\forall x \in \mathbf{R}, x^2 \geq 0.$

(b) $\forall x \in \mathbb{R}, x^2 \neq -1.$

(c) $\exists m \in \mathbb{Z}^+, m^2 = m.$

Solution:

(a) All real numbers have nonnegative squares.
Or: Every real number has a nonnegative square.
Or: Any real number has a nonnegative square.
Or: The square of each real number is nonnegative.

(b) All real numbers have squares that are not equal to −11.
Or: No real numbers have squares equal to −1.
(The words it none are or *no* .... are are equivalent to the words *all are not.*)

(c) There is a positive integer whose square is equal to itself.
Or: We can find at least one positive integer equal to its own square.
Or: Some positive integer equals its own square.
Or: Some positive integers equal their own squares.

Example 4.2. Translating from Informal to Formal Language Rewrite each of the following statements formally. Use quantifiers and variables.

(a) All triangles have three sides.

(b) No dogs have wings.

(c) Some programs are structured.

Solution:

(a) Suppose T is the set of all triangles.

(b) Suppose D is the set of all dogs.

(c) Suppose P is the set of all programs.

5 Equivalent Forms of Universal and Existential Statements

5.1 Equivalent form of Universal Conditional Statement

Consider the following universal conditional statement:

$$\forall x \in U, \text{if } \ P(x), then \ \ Q(x)$$

We will consider two equivalent forms.

5.1.1 Informal conditional statement without quantifiers or variables

It is common, as shown in the following example, to omit explicit identification of the domain U of predicate variables in universal conditional statements $\forall x \in U, \text{if } \ P(x), then \ \ Q(x)$.

Example 5.1. The following formal statement and informal statement (**without quantifiers or variables**) are equivalent.

(a)

- Formal form: $\forall x \in \mathbb{R}$, if $x > 2$, $x^2 > 4$.

  Informal forms without quantifiers or variables:

- If a real number is greater than 2 then its square is greater than 4.
- Whenever a real number is greater than 2, its square is greater than 4.
- The square of any real number greater than 2 is greater than 4.
- The squares of all real numbers greater than 2 are greater than 4.

(b)

- Formal form: $\forall x \in \mathbb{R}$, if $x \in \mathbb{Z}$, then $x \in \mathbb{Q}$
- Informal form:

(c)

- Formal form: $\forall x$, if x is a fire truck, then x is not green.
- Informal form:

5.1.2 Universal with unconditional statement

The following statements are equivalent.

- $\forall x \in U, \text{if } \ P(x), then \ \ Q(x)$
- $\forall x \in D$, $Q(x)$ where $D = \{x \in U, P(x) \text{ is true.}\}$

  I.e. the domain $D \subseteq U$ consists of all values of the variable $x \in U$ that make $P(x)$ true.
- $\forall x, \text{if } \ x \in D, then \ \ Q(x)$

Example 5.2. The following statements are equivalent.

- Informal form: All squares are rectangles.
- Formal form (Conditional statement): $\forall x$, if x is a square, then x is a rectangle.
- Formal form (Unconditional statement): ....

5.2 Equivalent form of Existential Statement

Similarly, a statement of the form

$$\exists x \in U,\ P(x)\ \wedge\ Q(x)$$

can be rewritten as

$$\exists x \in D,\ Q(x), \qquad \text{where } D = \{x \in U, P(x) \text{ is true}\}$$

or

$$\exists x \in \hat{D},\ P(x), \qquad \text{where } \hat{D} = \{x \in U, Q(x) \text{ is true}\}\ .$$

Example 5.3. Consider the statement

"There is an integer that is both prime and even."

Let Prime(n) be "n is prime and Even(n) be "n is even." The statement above is equivalent to:
- $\exists n$ such that Prime(n) $\wedge$ Even(n).
- $\exists$ a prime number n such that Even(n).
- $\exists$ an even number n such that Prime(n).

6 Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the universal conditional statement:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Notations
Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boxed{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently,

$$\forall x, P(x) \to Q(x).$$

- The notation $\boxed{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently,

$$\forall x, P(x) \leftrightarrow Q(x).$$

Example 6.1. Let the domain of x be the set of all positive integers $\mathbb{Z}^+$.
Consider the two predicates
$P(x)$: x is a factor of 4, and
$Q(x)$: x is a factor of 8.
Show that $P(x) \Rightarrow Q(x)$.
Solution:

7 Negation of Quantified Statements

Theorem 7.1 (Negation of a Universal Statement)**.** The *negation* of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D, \sim Q(x).$$

Symbolically,

$$\sim (\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x).$$

The negation of an existential statement is logically equivalent to a universal statement. E.g. the negation of the statement "some are ...." will be in the form "none are...." or "all are not..."

Example 7.1. Suppose the statement

All mathematicians wear glasses

is false. So a correct negation is its negation :

There is at least one mathematician who does not wear glasses.

Example 7.2.

Theorem 7.2 (Negation of an Existential Statement)**.** The *negation* of a statement of the form

$$\exists x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim Q(x).$$

Symbolically,

$$\sim (\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x).$$

The negation of an existential statement is logically equivalent to a universal statement. E.g. the negation of statement "some are...." will be in the form "none are..." or "all are not ... ."

Example 7.3. Rewrite the following statement formally. Then write formal and informal negations.

No politicians are honest.

Solution:

- Formal version:
- Formal negation:
- Informal negation:

Example 7.4. Write formal negations for the following statements:

- $\forall$ primes p, p is odd.
- $\exists$ a triangle T such that the sum of the angles of T equals 200°.

Solution:

Negation of a Universal Conditional Statement

$$\sim (\forall x, P(x) \rightarrow Q(x)) \equiv \exists x, P(x) \wedge \sim Q(x)$$

Example 7.5. Write the negation of each of the following statements:

i $\forall x \in \mathbb{R}; x > 3 \rightarrow x^2 > 9$

ii Every polynomial function is continuous.

iii There exists a triangle with the property that the sum of angles is greater than $180°$:

Solution:

The definitions of necessary, sufficient, and only if can also be extended to apply to universal conditional statements.

Example 7.6 (*necessary, sufficient, and only if*)**.**

- $\forall x, r(x)$ is a sufficient condition for $s(x)$, means $\forall x, r(x) \to s(x)$
- $\forall x, r(x)$ is a necessary condition for s(x) means $\forall x, \sim r(x) \to \sim s(x)$ or $\forall x, s(x) \to r(x)$.
- $\forall x, r(x)$ only if $s(x)$ means $\forall x, \sim s(x) \to \sim r(x)$ or, equivalently, $\forall x, r(x) \to s(x)$.

Definition 7.1. Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

- Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
- Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
- Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example 7.7. Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Solution: - The formal version of this statement is:

- Contrapositive:
Formal version:

Informal version: If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.
- Converse:
Formal version:

Informal version: If the square of a real number is greater than 4, then the number is greater than 2.
- Inverse:
Formal version:

Informal version:

8 Statements with Multiple Quantifiers

Consider the following statement:

" There is a person supervising every detail of the production process."

This statement can be considered as *ambiguous*, because it can be interpreted as, either one of these two different statements.

- "There is one single person who supervises all the details of the production process."
- "For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details."

For formal logic in mathematics, it is essential that we all interpret statements in exactly the same way. This can be done by using **multiple quantifiers** with certain systematic ways of interpreting statements as given below.

Interpreting Statements with Two Different Quantifiers

- To determine the truth of a statement of the form

$$\boxed{\forall x \in D \;\; \exists y \in E, P(x, y)} \text{ or}$$

$$\forall x \in D, \exists y \in E \;\text{ such that }\; P(x, y),$$

  we have to show that:
  for whatever element x in D is chosen, we must find an element y in E that "works" (i.e. $P(x, y)$ is true) for that particular x.

- To determine the truth of a statement of the form

$$\boxed{\exists x \in D \;\; \forall y \in E, \; P(x, y)} \text{ or}$$

$$\exists x \in D \;\text{ such that }\; \forall y \in E, \; P(x, y),$$

  we have to *find one particular x in D that will "work" (i.e. $P(x, y)$ is true) for every element y in E.*

Example 8.1. Determine the truth value of the statement

$$\forall x \in \mathbb{R} \; \exists y \in \mathbb{R}, \quad x + y = y + x.$$

Solution: Since "$x + y = y + x$" is true for any $x, y \in \mathbb{R}$, then this statement is always true. ■

Example 8.2. Let $P(x, y)$ denote the statement "$xy = 1$," where
the domain of x is the set of positive integers $\mathbb{Z}^+$ and
the domain of y is the set of all real numbers $\mathbb{R}$.
Determine the truth values of the following statements.

(i) For every positive integer x and for every real number y, "$xy = 1$. "

$$\forall x \in \mathbb{Z}^+ \ \forall y \in \mathbb{R}, P(x, y)$$

(ii) For every positive integer x there is a real number y such that $xy = 1$.

$$\forall x \in \mathbb{Z}^+ \ \exists y \in \mathbb{R}, P(x, y)$$

(iii) There exists a real number y such that, for every positive integer x, $xy = 1$.

$$\exists y \in \mathbb{R} \ \forall x \in \mathbb{Z}^+, P(x, y)$$

Solution:

Example 8.3. Let $Q(x, y)$ denote the statement "$x + y = 0$." Determine the truth values of the statements.

(i) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ \ Q(x, y)$ (ii) $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R},\ \ Q(x, y)$

Solution:

Example 8.4. Let $R(x, y)$ be the predicate "x understands y," where
the domain of x is the set of students in this TU152 class and
the domain of y is the set of examples in these lecture notes. Write the following statements using the quantifiers $\forall$, $\exists$, and the predicate $R(x, y)$.

(1) There exists a student in this class who understands every example in these lecture notes.

Answer:

(2) For every example in these lecture notes there is at least one student in the class who understands that particular example (it is possible that different students understand different examples).

Answer:

(3) Every student in this class understands at least one example in these notes.

Answer:

(4) There is an example in these notes that every student in this class understands.

Answer:

$$\exists x \forall y, R(x, y) \not\equiv \forall y \exists x, R(x, y) \qquad \text{and} \qquad \forall x \exists y, R(x, y) \not\equiv \exists y \forall x, R(x, y)$$

8.1 Negations of Multiply-Quantified Statements

Recall that, if we let $Q(x)$ be a predicate and D be the domain of x, then

$$\boxed{\sim \Big(\forall x, Q(x)\Big) \equiv \exists x, \sim Q(x)} \quad \text{and} \quad \boxed{\sim \Big(\exists x, Q(x)\Big) \equiv \forall x, \sim Q(x)}.$$

These can be extended to the negation of statements with multiple quantifiers as shown next.

Negations of Multiply-Quantified Statements
Let $P(x, y)$ be a predicate with two variables x and y.

- $$\sim \Big(\forall x \in D,\ \exists y \in E, P(x,y)\Big) \equiv \exists x \in D, \forall y \in E, \sim P(x,y)$$
- $$\sim \Big(\exists x \in D,\ \forall y \in E, P(x,y)\Big) \equiv \forall x \in D, \exists y \in E, \sim P(x,y)$$

In general, if we explicitly define D to be the domain of x and E to be the domain of y, then we can also write:

$$\boxed{\sim \Big(\forall x \exists y, P(x,y)\Big) \equiv \exists x \forall y, \sim P(x,y)}$$

and

$$\boxed{\sim \Big(\exists x \forall y, P(x,y)\Big) \equiv \forall x \exists y, \sim P(x,y)}$$

Simple Derivation:

- Show that $\boxed{\sim \Big(\forall x \exists y, P(x,y)\Big) \equiv \exists x \forall y, \sim P(x,y)}$.

From $\sim \Big(\forall x, Q(x)\Big) \equiv \exists x, \sim Q(x)$, and $\sim \Big(\exists x, Q(x)\Big) \equiv \forall x, \sim Q(x)$

$$\begin{aligned} \sim \Big(\forall x \exists y, P(x,y)\Big) &\equiv \sim \Big(\forall x \big(\exists y, P(x,y)\big)\Big) \\ &\equiv \exists x, \sim \Big(\exists y, P(x,y)\Big) \\ &\equiv \exists x \forall y, \sim P(x,y) \end{aligned}$$

■

- Show that $\boxed{\sim \Big(\exists x \forall y, P(x,y)\Big) \equiv \forall x \exists y, \sim P(x,y)}$.

From $\sim \Big(\forall x, Q(x)\Big) \equiv \exists x, \sim Q(x)$, and $\sim \Big(\exists x, Q(x)\Big) \equiv \forall x, \sim Q(x)$

$$\begin{aligned} \sim \Big(\exists x \forall y, P(x,y)\Big) &\equiv \sim \Big(\exists x \big(\forall y, P(x,y)\big)\Big) \\ &\equiv \end{aligned}$$

■

Example 8.5. Let D_x, D_y, and D_z be the domains for x, y, and z, respectively. Express the negations of the statement:

$$\forall x \exists y \forall z, T(x, y, z)$$

so that all negation symbols $\sim$ precede predicates.
Answer:

$$\begin{aligned} \sim \Big(\forall x \exists y \forall z, T(x, y, z)\Big) &\equiv \ \sim \forall x \Big(\exists y \forall z, T(x, y, z)\Big) \\ &\equiv \end{aligned}$$

Example 8.6. Let D_x, D_y be the domains for x, y, respectively. Express the negations of the statement:

$$\forall x \exists y, P(x, y) \ \vee \ \forall x \exists y, Q(x, y)$$

so that all negation symbols $\sim$ precede predicates.
Answer:

$$\begin{aligned} \sim \Big(\forall x \exists y, P(x, y) \ \vee \ \forall x \exists y, Q(x, y)\Big) &\equiv \\ &\equiv \end{aligned}$$

9 Arguments with Quantified Statements

In order to evaluate the validity of quantified statements, consider first the rule of **universal instantiation** which is given below.

> **The rule of universal instantiation:**
> "If some property is true of everything in a set, then it is true of any particular thing in the set."

- Universal instantiation is the fundamental tool of deductive reasoning.
- Mathematical formulas, definitions, and theorems are like general templates that are used over and over in a wide variety of particular situations.
- The term " universal instantiation " implies that *when a general (universal) case is true, a special case (certain instance)of this general case has to be true as well.*
  E.g. let D be the domain of variable x in the predicate $Q(x)$.

  - General (universal) case: Suppose $Q(x)$ is true for any $x \in D$.
  - Special case (certain instance): Suppose $a \in D$. $\Rightarrow$ $Q(a)$ is true.
- A famous example:

  All human beings are mortal.

  Socrates[1] is a human being.

  $\therefore$ Socrates is mortal.

  Note that we can write the above argument in the following forms.
  Let $P(x)$ be "x is a human being" and $Q(x)$ be "x is mortal."

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x is a human being, then x is mortal.
$P(Socrates)$ is true.	Socrates is a human.
$\therefore Q(Socrates)$ is true.	$\therefore$ $x = Socrates$ makes $Q(x)$ true.

- The rule of universal instantiation can be combined with rules of inferences to form a **valid** arguments with quantifiers. Rules of inferences that will be considered are
  - Modus Ponens
  - Modus Tollens
  - Transitivity.

[1] A classical Greek philosopher credited as one of the founders of Western philosophy.

9.1 Universal Modus Ponens

When the rule of **universal instantiation** is combined with **modus ponens** to obtain a valid form of argument, this combination is called **universal modus ponens**.

Universal Modus Ponens

Formal Version	**Informal Version**
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore Q(a)$.	$\therefore$ a makes $Q(x)$ true.

Example 9.1. Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

k is a particular integer that is even.

$\therefore$ k^2 is even.

Solution:

9.2 Universal Modus Tollens

When the rule of **universal instantiation** is combined with **modus tollens** to obtain a valid form of argument, this combination is called **universal modus tollens**. Universal modus tollens is the main concept of *proof by contradiction*, which is one of the most important methods of mathematical argument.

Universal Modus Tollens

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$ for a particular a.	a does not make $Q(x)$ true.
$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Example 9.2. Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All lawyers went to law schools.

Tom didn't go to a law school.

$\therefore$ Tom is not a lawyer.

Solution:

9.3 Universal Transitivity

Universal Transitivity

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\forall x, Q(x) \to R(x)$.	If x makes $Q(x)$ true, then x makes $R(x)$ true.
$\therefore \forall x, P(x) \to R(x)$.	$\therefore$ If x makes $P(x)$ true, then x makes $R(x)$ true.

Example 9.3. Show that the following argument is valid by **universal modus ponens** , **universal modus tollens** and/or **universal transitivity**.
"Anyone who has a school email account has a school ID number."
"All students have school email accounts."
"Kim does not have a school ID number."
$\therefore$ "Kim is not a student."

Solution: Let
$P(x)$ be "x has a school email account,"
$Q(x)$ be "x has a school ID number,"
$R(x)$ be "x is a student."
Then we can transform the given argument in the quantified form as follows.

and using the universal transitivity rule we have

Using the result from the transitivity rule above, $\forall x, R(x) \to Q(x)$, the original argument becomes

which is valid by.................................
That is, the original argument is **vilid** by **the universal transitivity and universal modus tollens**.
■

9.4 Converse & Inverse Errors (Quantified Form)

The following forms of arguments are **invalid**.

Converse Error (Quantified Form)

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$.	$\therefore$ a makes $P(x)$ true.

Inverse Error (Quantified Form)

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$.	$\therefore$ a does not make $Q(x)$ true.

Example 9.4. Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All lawyers went to law schools.

Tom is not a lawyer.

$\therefore$ Tom didn't go to a law school.

Solution:

9.5 Validity test for arguments with quantified statements by diagrams

The intuitive definition of validity for arguments with quantified statements is the same as for arguments with compound statements. An argument is valid if, and only if, the truth of its conclusion follows necessarily from the truth of its premises. The formal definition is as follows:

Definition 9.1.

- To say that an argument form is **valid** means the following:

  No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

- An argument is called **valid** if, and only if, its form is valid.
- Note: the truth of premises is **not** relevant to the validity of the arguments.

9.5.1 Using Diagrams to Test for Validity

The concept of using diagrams to analyze arguments was initially used by Gottfried Wilhelm Leibniz, a German mathematician and philosopher. [2]

Using diagrams to analyze the validity or invalidity of arguments that contain quantified statements can be really helpful to make it easier to understand. However, diagrams do not provide totally rigorous proofs of validity and invalidity and may not be suitable for complicated problems.
To test the validity of an argument diagrammatically,

- represent the truth of both premises with diagrams, and
- analyze the diagrams to see whether they necessarily represent the truth of the conclusion also.

Example 9.5. Let the set of all BE students be the domain D. Suppose Jane is a BE student.
Use a diagram to show the invalidity of the following argument.

All freshmen must take TU152.

Jane takes TU152.

$\therefore$ Jane is a freshman.

Solution:

.

[2] Leibniz also developed the main ideas of the differential and integral calculus at approximately the same time as (and independently of) Isaac Newton (1642- 1727).

Example 9.6. Use a diagram to show the invalidity of the following argument.

All lawyers went to law schools.

Tom went to a law school.

$\therefore$ Tom is a lawyer.

Solution:

Example 9.7. Use a diagram to show the validity of the following argument.

All lawyers went to law schools.

Tom didn't go to a law school.

$\therefore$ Tom is not a lawyer.

Solution:

Example 9.8. (Exercise) Use a diagram to show the invalidity of the following argument:

No politicians are honest.

Jane is honest.

$\therefore$ Jane is not a politician.

<u>Solution:</u>

<u>Exercise</u> Rewrite the following arguments using quantifiers, variables, and predicate symbols. Determine if these arguments are valid. Explain your answers.

(a)

All human beings are mortal.

Buster is mortal.

$\therefore$ **Buster** is a human being.

(b)

Any sum of two rational numbers is rational.

The sum $r + s$ is rational.

$\therefore$ The numbers r and s are both rational.

(c)

All freshmen must take TU152.

Jane is a freshman.

$\therefore$ Jane must take TU152.

(d)

All healthy people eat an apple a day

Jane eats an apple a day.

$\therefore$ Jane is a healthy person.