Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean $(\mu) = 0$ and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

$1 - 1.5B + 0.9B^2$

→ Reverse Characteristics

: $x^2 - 1.5x + 0.9$ (ax^2+bx+c)

using $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(1)(0.9)}}{2(1)}$

$= \frac{3}{4} + \frac{3\sqrt{15}}{20}i \, , \, \frac{3}{4} - \frac{3\sqrt{15}}{20}i$

Test Modulus: $\sqrt{\left(\frac{3}{4}\right)^2 + \left(-\frac{3\sqrt{15}}{20}\right)^2} = 0.949 < 1$

$\sqrt{\left(\frac{3}{4}\right)^2 + \left(-\frac{3\sqrt{15}}{20}\right)^2} = 0.949 < 1$

∴ R_t is weak stationarity

Question 1.2 (10 points) Your score......................

Given $\varepsilon_t \overset{iid}{\sim} N(0, \sigma_a^2 = 0.25)$, R_t is weak stationary, for $\forall t$:

① $E[R_t] = \mu$

② $var[R_t] = \sigma_R^2$

③ $cov[R_t, R_{t-j}] = \gamma_j$

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

$R_t = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + \varepsilon_t$

Unconditional Mean:

$E(R_t) = E\left[0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + \varepsilon_t\right]$

$\mu = 0.25 + 1.5 E[R_{t-1}] - 0.9 E[R_{t-2}] + E[\underbrace{\varepsilon_t}_{0 \text{ ~ white noise}}]$

$\mu = 0.25 + 1.5\mu - 0.9\mu$

$\mu - 1.5\mu + 0.9\mu = 0.25$

$\mu(1 - 1.5 + 0.9) = 0.25$

$\mu = \frac{0.25}{(1-1.5+0.9)} = \frac{5}{8} \rightarrow 0.4\mu = 0.25$

Conditional Mean:

Given $F_{t-1} = \{R_{t-1}, R_{t-2}, \dots, R_2, R_1\}$

$E(R_t | F_{t-1}) = E\left[0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + \varepsilon_t \mid F_{t-1}\right]$

$= E[0.25 | F_{t-1}] + E[1.5 R_{t-1} | F_{t-1}] - E[0.9 R_{t-2} | F_{t-1}] + E[\underbrace{\varepsilon_t | F_{t-1}}_{0 \text{ ~ white noise}}]$

$E[R_t | F_{t-1}] = 0.25 + 1.5 E[R_{t-1} | F_{t-1}] - 0.9 E[R_{t-2} | F_{t-1}]$

$E[R_t | F_{t-1}] = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2}$

Question 1.3 (10 points) Your score......................

$$\mu = \frac{0.25}{(1-1.5+0.9)} = \frac{5}{8} \longrightarrow 0.4\mu = 0.25$$

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$$

unconditional :

$$R_t = 0.4\mu + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$$

$$R_t = 0.4\mu \underline{-1.5\mu + 1.5\mu} + 1.5R_{t-1} - 0.9R_{t-2} \underline{+0.9\mu - 0.9\mu} + \varepsilon_t$$

$$R_t - \mu = 0.4\mu - \mu + 1.5\mu + 1.5(R_{t-1}-\mu) - 0.9(R_{t-2}-\mu) - 0.9\mu$$

$$(R_t-\mu) = (0.4-1+1.5-0.9)\mu + 1.5(R_{t-1}-\mu) - 0.9(R_{t-2}-\mu) + \varepsilon_t$$

Demean : $$(R_t-\mu) = 1.5(R_{t-1}-\mu) - 0.9(R_{t-2}-\mu) + \varepsilon_t$$

$$(R_t-\mu)^2 = 1.5^2(R_{t-1}-\mu)^2 + (-0.9)^2(R_{t-2}-\mu)^2 + \varepsilon_t^2 + 2(1.5)(-0.9)(R_{t-1}-\mu)(R_{t-2}-\mu) + 2(1.5)(R_{t-1}-\mu)(\varepsilon_t) + 2(-0.9)(R_{t-2}-\mu)(\varepsilon_t)$$

$$\underbrace{E(R_t-\mu)^2}_{var(R_t)=\sigma_R^2} = 1.5^2 \underbrace{E[(R_{t-1}-\mu)^2]}_{\sigma_R^2} + (-0.9)^2 \underbrace{E[(R_{t-2}-\mu)^2]}_{\sigma_R^2} + \underbrace{E(\varepsilon_t^2)}_{\sigma_a^2=0.25} + 2(1.5)(-0.9)\underbrace{E[(R_{t-1}-\mu)(R_{t-2}-\mu)]}_{\gamma_1} + 2(1.5)\underbrace{E[(R_{t-1}-\mu)\varepsilon_t]}_{0} + 2(-0.9)\underbrace{E[(R_{t-2}-\mu)\varepsilon_t]}_{0}$$

$$\sigma_R^2 = 2.25\sigma_R^2 + 0.81\sigma_R^2 + 0.25 - 2.7\gamma_1$$

$$\sigma_R^2(1-2.25-0.81) = 0.25 - 2.7\gamma_1$$

$$\sigma_R^2 = \frac{0.25-2.7\gamma_1}{-2.06} = \gamma_0 \ \#; \quad \gamma_1 = \rho_1\gamma_0 = \frac{15}{19}\gamma_0 \longrightarrow -\frac{0.25}{2.06} + \frac{2.7}{2.06}\left(\frac{15}{19}\right)\gamma_0 = \gamma_0$$

$$\gamma_0 = 3.493 \ \#$$

Conditional :

$$var(R_t|\cdot) = var(0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t|\cdot)$$

$$= var(\cancelto{0}{0.25}|\cdot) + 1.5^2 var(\cancelto{0}{R_{t-1}}|\cdot) + 0.9\, var(\cancelto{0}{R_{t-2}}|\cdot) + var(\varepsilon_t|\cdot) + cov(\cancelto{0}{R_{t-1}}, \varepsilon_t|\cdot) + cov(\cancelto{0}{R_{t-2}}, \varepsilon_t|\cdot) + cov(\cancelto{0}{R_{t-1}, R_{t-2}}|\cdot)$$

$$= var(\varepsilon_t|\cdot)$$

$$= \sigma_a^2 = 0.25 \ \#$$

Question 1.4 (10 points)

Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

Demean : $(R_t - \mu) = 1.5(R_{t-1} - \mu) - 0.9(R_{t-2} - \mu) + \varepsilon_t$

$(R_t-\mu)(R_{t-j}-\mu) = 1.5(R_{t-1}-\mu)(R_{t-j}-\mu) - 0.9(R_{t-2}-\mu)(R_{t-j}-\mu) + \varepsilon_t(R_{t-j}-\mu)$

$E[(R_t-\mu)(R_{t-j}-\mu)] = 1.5E[(R_{t-1}-\mu)(R_{t-j}-\mu)] - 0.9E[(R_{t-2}-\mu)(R_{t-j}-\mu)] + \underbrace{E[\varepsilon_t(R_{t-j}-\mu)]}_{0}$

$\gamma_j = 1.5\gamma_{j-1} - 0.9\gamma_{j-2}$

divide by γ_0 : $\frac{\gamma_j}{\gamma_0} = 1.5\frac{\gamma_{j-1}}{\gamma_0} - 0.9\frac{\gamma_{j-2}}{\gamma_0}$; $\rho_j = \frac{\gamma_j}{\gamma_0}$

$\rho_j = 1.5\rho_{j-1} - 0.9\rho_{j-2}$

let $j=1$,

$\rho_1 = 1.5\rho_0 - 0.9\rho_{-1}$; $\rho_{-1} = \rho_1$ from stationarity and $\rho_0 = 1$

$\rho_1 = 1.5 - 0.9\rho_1$

$\rho_1 = \frac{1.5}{1.9} = \frac{15}{19}$ #

and

$j=2$, $\rho_2 = 1.5\rho_1 - 0.9\rho_0 = 1.5\left(\frac{15}{19}\right) - 0.9 = \frac{27}{95}$ #

$j>2$; $\rho_j = 1.5\rho_{j-1} - 0.9\rho_{j-2}$ given that $\rho_1 = \frac{15}{19}$, $\rho_2 = \frac{27}{95}$ #

Question 1.5 (10 points) Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

$R_t = 0.25 + 1.5 R_{t-1} - 0.9 R_{t-2} + \varepsilon_t$; at $t = 1000$

1-step : $R_{1001} = 0.25 + 1.5 R_{1000} - 0.9 R_{999} + \varepsilon_{1001}$

$E[R_{1001} | F_{1000}] = \hat{R}_{1000}(1) = E[0.25|\cdot] + E[1.5 R_{1000}|\cdot] + E[-0.9 R_{999}|\cdot] + \overset{0}{E[\varepsilon_{1001}|\cdot]}$

$\therefore \hat{R}_{1000}(1) = 0.25 + 1.5(0.01) - 0.9(0.02) = 0.247$

$R_{1001} - \hat{R}_{1000}(1) = \varepsilon_{1001}$ and $var(\varepsilon_{1001}|\cdot) = \sigma_a^2 = 0.25$

$\therefore$ 95% interval is $\hat{R}_{1001}(1) \pm Z_{\frac{0.05}{2}} \sqrt{\sigma_a^2} = 0.247 \pm 1.96\sqrt{0.25}$

$= [-0.733, 1.227]$ 1-step 95% interval forecast for R_{1001} at $t=1000$

2-step : $R_{1002} = 0.25 + 1.5 R_{1001} - 0.9 R_{1000} + \varepsilon_{1002}$

$E[R_{1002}|\cdot] = \hat{R}_{1000}(2) = E[0.25|\cdot] + 1.5\hat{R}_{1000}(1) - 0.9 E[R_{1000}|\cdot] + E[\varepsilon_{1002}|\cdot]$

$= 0.25 + 1.5(0.247) - 0.9(0.01) = 0.6115$

$e_{1000}(2) = R_{1002} - \hat{R}_{1000}(2) = 1.5[R_{1001} - \hat{R}_{1000}(1)] + \varepsilon_{1002} = 1.5\varepsilon_{1001} + \varepsilon_{1002}$

$var(e_{1000}(2)) = 1.5^2 var(\varepsilon_{1001}|\cdot) + var(\varepsilon_{1002}|\cdot) + \overset{0}{2cov(\ )} = 1.5^2\sigma_a^2 + \sigma_a^2 = 0.8125$

$\therefore$ 95% interval is $\hat{R}_{1001}(2) \pm Z_{\frac{0.05}{2}}\sqrt{0.8125} = 0.6115 \pm 1.96\sqrt{0.8125}$

$= [-1.552, 2.3782]$ 1-step 95% interval forecast for R_{1002} at $t=1000$

∞-step

$\lim_{l\to\infty} \hat{R}_{1000}(l) = E[r_t] = \frac{5}{8}$

$var(e_{1000}(\infty)) = var(r_t) = 3.493$