B.E. International Program

Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics

Semester 2/2013

Practice Problem 6

(Optimization problem – 1 independent variable)

Suggested Answers

1. Find the local extreme points for:

a) $f(x) = x^3 e^x$

Ans. $f'(x) = x^2 e^x (x + 3)$. $x^* = 0, -3$. f(0) is an inflection point; f(-3) is a minimum.

b) $g(x) = x^2 2^x$

Ans. $\frac{d(2^x)}{dx} = 2^x ln2$; $g'(x) = x^2 \cdot 2^x ln2 + 2^x(2x)$

Critical values: x*= 0, -2/ln2;

f''(0) > 0 ➔ *f(0) is a local minimum.*

f''(-2/ln2) < 0 ➔ *f(-2/ln2) is a local maximum.*

2. Exercise 9.2 question 1 in Chiang and Wainwright (2005)

Ans. (a) $f'(x) = -4x + 8 = 0$ iff $x = 2$; the stationary value $f(2) = 15$ is a relative maximum.

(b) $f'(x) = 10x + 1 = 0$ iff $x = -1/10$; $f(-1/10)=-1/20$ is relative minimum.

(c) $f'(x) = 6x = 0$ iff $x = 0$; $f(0) = 3$ is a relative minimum.

3. Exercise 9.4 questions 3 and 6 in Chiang and Wainwright (2005)

Ans. Q. 3. (a) Yes.

(b) AR function: P = 100 – Q. Thus, $R = PQ = (100 - Q)Q = 100Q - Q^2$.

(c) $\pi = R - C = \frac{1}{3}Q^3 + 6Q^2 - 11Q - 50$

(d) Set $\frac{d\pi}{dQ} = -Q^2 + 12Q - 11 = 0$. Then, only Q* = 11 gives a maximum profit.

(e) Maximum profit = 111.33

Q.6 (a) $Q = f(L)$; $R = P_0 Q = P_0 f(L)$; $C = W_0 L + F$; $\pi = P_0 f(L) - W_0 L - F$

(b) $\frac{d\pi}{dL} = P_0 f'(L) - W_0 = 0$ or $P_0 f'(L) = W_0$. The value of marginal product must be equal to the wage rate.

(c) $\frac{d^2\pi}{dL^2} = P_0 f''(L)$. If $f''(L) < 0$, then profit is maximized at L*.

4. For the following functions, (i) find the critical values, (ii) test for concavity to determine relative maxima or minima, (iii) check for inflection points, and (4) evaluate the function at the critical value and inflection points.

a) $f(x) = x^3 - 18x^2 + 96x - 80$

Ans. Critical values: x = 4 (relative maximum), x = 8 (relative minimum);
Inflection point: x =6

b) $f(x) = -x^3 + 6x^2 + 15x - 32$

Ans. critical values: x = -1 (relative minimum), x = 5 (relative maximum);
Inflection point: x =2

c) $f(x) = (2x - 7)^3$

Ans. critical values: x = 3.5;
Test for concavity is inconclusive.

(Questions 5 – 7 are from Sydsaeter and Hammond, 2008)

5. Suppose that

$$R(Q) = 10Q - \frac{Q^2}{1000}, \qquad C(Q) = 5000 + 2Q, \qquad Q \in [0, 10{,}000]$$

Find the value of Q that maximizes profits.

Ans. Q* = 4000

6. The price a firm obtains for a commodity varies with demand Q according to the formula $P(Q) = 18 - 0.006Q$. Total cost is $C(Q) = 0.004Q^2 + 4Q + 4500$.

a. Find the firm's profit and the value Q which maximizes profit.

Ans. $\pi(Q) = Q \cdot P(Q) - C(Q) = -0.01Q^2 + 14Q - 4500$. Q* = 700.

b. Find a formula for the elasticity of $P(Q)$ w.r.t. Q, and find the particular value Q^* of Q at which the elasticity is equal to -1.

Ans. This is the price elasticity with respect to demand (Q refers to quantity demanded).

$$E_d = \frac{Q}{P} \cdot \frac{dP}{dQ} = \frac{Q}{18 - 0.006Q} \cdot (-0.006) = -1 \text{ for Q* = 1500.}$$

c. Show that the marginal revenue is 0 at Q*.

Ans. $(Q) = Q \cdot P(Q) = 18Q - 0.006Q^2$. $MR = R'(Q) = 18 - 0.012Q = 0$ for Q* = *1500.*

7. A competitive firm receives a price p for each unit of its output, and pays a price w for each unit of its only variable input. It also incurs set up costs of F. Its output from using x units of variable input is $f(x) = \sqrt{x}$. Determine the firm's revenue, cost, and profit functions.

Ans. For x > 0, $R = p\sqrt{x}$, $C = wx + F$, $\pi(x) = p\sqrt{x} - wx - F$.

$\pi'(x) = 0$ when $w = p/2\sqrt{x}$. ➔ $x = p^2/4w^2$. Check $\pi''(x) < 0$.

8. Given the following function

$$f(x) = 2x^3 + 8x^2 - 32x - 50,$$

a) Find the critical value(s) of x and the corresponding stationary value(s) of $f(x)$.

Ans. $f'(x) = 6x^2 + 16x - 32 = 0 \quad \Leftrightarrow \; x^* = -4, \frac{4}{3}$

b) Evaluate whether the stationary value(s) found in part a) are relative maxima or minima or inflection points by using the first-derivative test.

Ans. f(-4) is a maximum because $f'(-5) = 38 > 0$; $f'(-3) = -26 < 0$.

f(3/4) is a minimum because $f'(1) = -10 < 0$; $f'(2) = 24 > 0$.

9. Let the total cost function be:

$$TC(Q) = 2Q^2 - 8Q + 10.$$

a) Determine whether $TC(Q)$ is a convex or concave function.

Ans. $TC'(Q) = 4Q - 8 \; \Rightarrow \; TC''(Q) = 4 > 0$. Thus, TC(Q) is a convex function.

b) Find the quantity Q^* that minimizes the total cost.

Ans. $TC'(Q) = 4Q - 8 = 0 \; \Rightarrow Q^* = 2$

c) Verify that $TC(Q^*)$ is the lowest cost by using the second derivative test.

Ans. $TC''(2) = 4 > 0$. Thus, TC(2) = 2 is the minimum.

10. Suppose that a firm produces a single commodity. Its demand and total-cost functions are:

$$Q = 70 - P$$

$$C(Q) = Q^2 - 10Q + 500.$$

a) Set up a profit function, $\pi(Q)$.

Ans. $\pi(Q) = (70 - Q)Q - (Q^2 - 10Q + 500) = -2Q^2 + 80Q - 500$

b) Find the value of Q^* that maximizes profit.

Ans. $\pi'(Q) = -4Q + 80 = 0 \; \Rightarrow Q^* = 20$

c) Verify that the quantity Q^* found in part (b) gives a maximum profit.

Ans. $\pi''(Q) = -4 < 0$. Thus, $\pi(20) = 300$ is the maximum profit.