Solution: Assignment 2

1. Let $P(x)$ be the predicate "$\frac{x}{(x+2)^3} < 0$." Find the *truth set* of $P(x)$ if the domain of predicate variable x is the set of:
(a) real numbers $\mathbb{R}$, (b) positive real numbers $\mathbb{R}^+$, (c) integers $\mathbb{Z}$.

Answer: Notice that if $x \in \mathbb{R}$, then two possibilities that can make $\frac{x}{(x+2)^3} < 0$ are:

(i) $x < 0$ and $x + 2 > 0$. This implies $x < 0$ and $x > -2$ or $\boxed{-2 < x < 0}$.

(ii) $x > 0$ and $x + 2 < 0$. This implies $x > 0$ and $x < -2$ which is **impossible**.

(a) When the domain is $\mathbb{R}$, the truth set is the open interval $(-2, 0)$.
(b) When the domain is $\mathbb{R}^+$, $P(x)$ is always false and hence the truth set is an empty set $\emptyset$.
(c) When the domain is $\mathbb{Z}$, $P(x)$ is only true when $x = -1$ and hence the truth set is $\{-1\}$

2. Let $Q(x, y)$ be the predicate "If $\frac{x}{y} < 1$ then $x^2 < y^2$" with domain for both x and y being the set $\mathbb{R}$ of real numbers. Determine the truth value of $Q(x, y)$ when
(a) $x = -3, y = 1$, (b) $x = 3, y = 1$.

Answer:

(a) $Q(-3, 1)$ is false, because $-3 < 1$ but $(-3)^2 \not< 1^2$ (true hypothesis, but false conclusion).
(b) $Q(3, 1)$ is true, because the hypothesis $3 < 1$ is false, which makes $Q(3, 1)$ always true (even with false conclusion or $(3)^2 \not< 1^2$).

3. Let the set $\mathbb{Z}^+$ of all positive integers be the domain of the predicate variable x.

(a) Find the truth set for each predicate: " $\sqrt{x} > 2$ " and "$x^2 > 23$. "

(b) Determine if the following statements (i) and (ii) are true or false. Give counterexamples for the statements that are false.
(i) $\sqrt{x} > 1 \Rightarrow x^2 > 23$, (ii) $\sqrt{x} > 2 \Leftrightarrow x^2 > 23$.

Answer:

(a) -Notice that $\sqrt{x} > 2$ only when $x > 4$.
So " $\sqrt{x} > 2$ " is true when $x \in \{5, 6, 7, 8, \dots\}$.
Hence, truth set for predicate: " $\sqrt{x} > 2$ " is $\{5, 6, 7, 8 \dots, \}$ or $\mathbb{Z}^+/\{1, 2, 3, 4\}$.
-Notice that $x^2 > 23$ only when $x > \sqrt{23}$ for $x \in \mathbb{Z}^+$.
So "$x^2 > 23$" is true when $x \in \{5, 6, 7, 8, \dots\}$.
Hence, truth set for predicate: "$x^2 > 23$" is $\{5, 6, 7, 8 \dots, \}$ or $\mathbb{Z}^+/\{1, 2, 3, 4\}$.

(b) (i) $\sqrt{x} > 1 \Rightarrow x^2 > 23$ is false since the truth set of "$x^2 > 23$" is not contained in the truth set of "$\sqrt{x} > 1$." In particular,

"$\sqrt{x} > 1$" is true when $x \in \{2, 3, 4, 5 \dots\}$ and hence its truth set is $T_1 := \{2, 3, 4, 5 \dots\}$.
"$x^2 > 23$" is true when $x \in \{5, 6, 7, 8 \dots, \}$ and hence its truth set is $T_2 := \{5, 6, 7, 8 \dots, \}$.

Since $T_1 \not\subseteq T_2$, this statement is false. Also, a counterexample is $x = 2$ or any integer $x \geq 2$. (Note: it is enough the just give a counterexample here).

(ii) $\sqrt{x} > 2 \Leftrightarrow x^2 > 23$ is true because, from (a), the truth sets of " $\sqrt{x} > 2$ " and "$x^2 > 23$" are the same.

4. Let D be the set of all students who are in a math class, and for $s \in D$,
let $M(s)$ be "s is a math major,"
let $C(s)$ be "s is a computer science student," and
let $E(s)$ be "s is an engineering student."
Express each of the following statements using quantifiers, variables, and the predicates $M(s)$, $C(s)$, and $E(s)$.

(a) There is an engineering student who is a math major.
(b) Every computer science student is an engineering student.
(c) No computer science students are engineering students.
(d) Some computer science students are also math majors.
(e) Some computer science students are engineering students and some are not.

Answer:

(a) There is an engineering student who is a math major: $\boxed{\exists s \in D \text{ such that } E(s) \wedge M(s).}$.
(b) Every computer science student is an engineering student: $\boxed{\forall s \in D \text{ such that } C(s) \rightarrow E(s)}$.
(c) No computer science students are engineering students: $\boxed{\forall s \in D \text{ such that } C(s) \rightarrow \sim E(s)}$.
(d) Some computer science students are also math majors: $\boxed{\exists s \in D \text{ such that } C(s) \wedge M(s)}$.
Alternative answer: $\exists s \in D$ such that $C(s) \rightarrow M(s)$
(e) Some computer science students are engineering students and some are not:

$$(\exists s \in D \text{ such that } C(s) \wedge E(s)) \wedge (\exists s \in D \text{ such that } C(s) \wedge \sim E(s)) .$$

5. Write a negation for each statement.

(a) $\exists x \in \mathbb{R}$, if $(x-1)(x+1) > 0$, then $x > 1$ or $x < -1$.
(b) $\forall$ integers a, b and c, if $a - b$ is even and $b - c$ is even, then $a - c$ is even.
(c) The limit of a real-valued function $f(x)$ at point a is L, $\lim_{x \to a} f(x) = L$, which is defined as

$$\forall \varepsilon \exists \delta \forall x \left(|x - a| < \delta \rightarrow |f(x) - L| < \varepsilon \right),$$

where the domain for ε, δ, x is the set of real numbers. I.e. Find the negation of $\lim_{x \to a} f(x) = L$ (which is"$\lim_{x \to a} f(x) \neq L$") in terms of multiple quantified statement.

Answer:

(a) $\sim (\exists x \in \mathbb{R} \text{ if } (x-1)(x+1) > 0, \text{ then } x > 1 \text{ or } x < -1).$
$\equiv \forall x \in \mathbb{R}, \sim [\text{ if } (x-1)(x+1) > 0, \text{ then } x > 1 \text{ or } x < -1]$
$\equiv \forall x \in \mathbb{R}, (x-1)(x+1) > 0 \text{ and } (x \leq 1 \wedge x \geq -1)$

(b) $\sim [\forall$ integers a, b and c, if $a-b$ is even and $b-c$ is even, then $a-c$ is even$]$
$\equiv \exists$ some integers a, b, c, $\sim$ [if $a-b$ is even and $b-c$ is even, then $a-c$ is even]
$\equiv \exists$ some integers a, b, c, $a-b$ is even and $b-c$ is even, but $a-c$ is odd.

(c) $\sim \forall\varepsilon\exists\delta\forall x\,(|x-a|<\delta \to |f(x)-L|<\varepsilon)$
$\equiv \exists\varepsilon \sim [\exists\delta\forall x\,(|x-a|<\delta \to |f(x)-L|<\varepsilon)]$
$\equiv \exists\varepsilon\forall\delta \sim [\forall x\,(|x-a|<\delta \to |f(x)-L|<\varepsilon)]$
$\equiv \exists\varepsilon\forall\delta\exists x \sim (|x-a|<\delta \to |f(x)-L|<\varepsilon)$
$\equiv \exists\varepsilon\forall\delta\exists x\,(|x-a|<\delta \;\wedge\; |f(x)-L|\geq\varepsilon)$

6. The notation $\exists!$ stands for the words "there exists a unique." Thus, for instance, "$\exists! x$ such that x is prime and x is even" means that there is **one and only one** even prime number. Which of the following statements are true and which are false? Explain.

(a) $\exists!$ real number x such that $\forall$ real numbers y, $xy = y$.

(b) $\exists!$ integer x such that $1/x$ is an integer.

(c) $\forall$ real numbers x, $\exists!$ real number y such that $x + y < 0$.

Answer:

(a) The statement is true. The unique real number with the given property is $x = 1$, since

$$1 \cdot y = y, \quad \text{for all real number } y.$$

(b) The statement is false, since both $x = -1$ and $x = 1$ (and hence x is not unique).

$$1/x = 1 \quad \text{is an integer}$$

i.e. $\frac{1}{-1} = -1$ and $\frac{1}{1} = 1$ are both integers.

(c) The statement is false. For each given real number x, there are **many values** of real number y (any $y < -x$ e.g.$y = -x - 1$ or $-x - 2$) that make $x + y < 0$. Hence y is not unique. For example, if $x = 3$, we can choose any $y < -3$ (e.g. $y = -4, -5, -6, -7, \ldots$), so that $x + y < 0$.

7. Show that each of the following arguments is valid by **universal modus ponens** ,**universal modus tollens**,**universal transitivity**, or show that it is invalid from the **converse error** or the **inverse error**. In addition, use also the **diagram** to confirm that each argument is valid or invalid.

(a) All honest people pay their taxes.
Darth is not honest.
$\therefore$ Darth does not pay his taxes.

(b) No vegetarians eat meat.
All vegans are vegetarians.
$\therefore$ No vegans eat meat.

Answer:

(a) We can transform the given argument in the quantified form of **inverse error:**

$$\forall x, P(x) \rightarrow Q(x)$$
$$\sim P(a) \text{ for a particular } a= \text{Darth}$$
$$\therefore \sim Q(a)$$

where $P(x)$ is defined as "x is honest" and $Q(x)$ is defined as "x does not pay tax." Hence the argument is **invalid**.

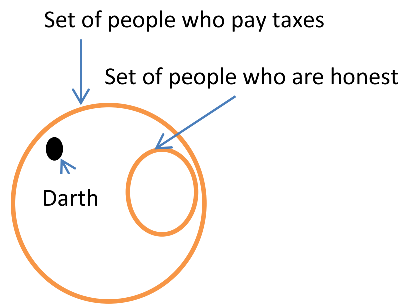

Figure 1: Problem 7 (a)

From the diagram, since it is possible that the "Dart " is still inside the set of people who pay taxes, even if "Dart " is not in the set of honest people, then the statement is **invalid**.

(b) First the given premise " No vegetarians eat meat" can be written in terms of *if-then statement* as "$\forall x$, if x is a vegetarian, then x does not eat meat."
Similarly, the given premise "All vegans are vegetarians" in the form of *if-then statement* as "$\forall x$, if x is a vegan, then x is a vegetarian."

Then, the given argument can be re-written in a valid form of **universal modus ponens**:

$$\forall x, P(x) \rightarrow Q(x)$$
$$P(a) \text{ for a particular } a = \text{vegan}$$
$$\therefore Q(a),$$

where $P(x)$ is defined as "x is a vegetarian" and $Q(x)$ is defined as "x does not eat meat." Hence the argument is **valid**.
Note: we can also use the universal transitivity in the following form:

$$\forall x, P(x) \rightarrow Q(x)$$
$$\forall x, Q(x) \rightarrow R(x)$$
$$\therefore \quad \forall x, P(x) \rightarrow R(x).$$

Diagram
From the diagram, since vegans are in the set of vegetarians, which is in the set of people who do not meat. Hence the conclusion that no vegans eat meat is **valid**.

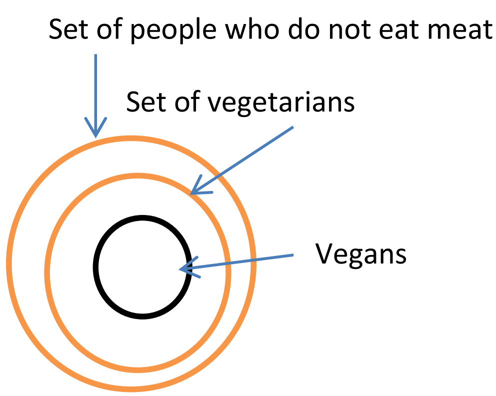

Figure 2: Problem 7 (b)