

EE320 (2/2012)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION WITHOUT CONSTRAINTS:

MORE-THAN-ONE-INDEPENDENT VARIABLE CASES

Topics

- The differential version of optimization conditions
- Two choice variable optimization
 - Conditions for maximum or minimum
 - Economics examples:
 - Multiplant-firm
 - Multiproduct-firm
 - Third degree price discrimination
- Multivariable optimization
 - Conditions for maximum or minimum
 - Economics examples

Differential Version of Optimization Conditions

- Given $z = f(x)$, the differential of z is:

$$dz = f'(x)dx, \quad \text{where } f'(x) = dz/dx.$$

- The **first-order condition (in terms of derivative)** for an optimum:

$$f'(x) = 0$$

- Graph:

- The **first-order differential condition** for any arbitrary $dx = 0$:

$$dz = 0.$$

Differential Version of Optimization Conditions

- The **second-order sufficient condition** (in terms of derivative):
 - For a maximum of z : $f''(x) < 0$.
 - For a minimum of z : $f''(x) > 0$.

- Rewrite the S.O.C. in terms of differential:

$$\begin{aligned}d^2z &= d(dz) = d[f'(x)dx] \\ &= [df'(x)]dx \\ &= [f''(x)dx]dx\end{aligned}$$

$$\rightarrow d^2z = f''(x)dx^2$$

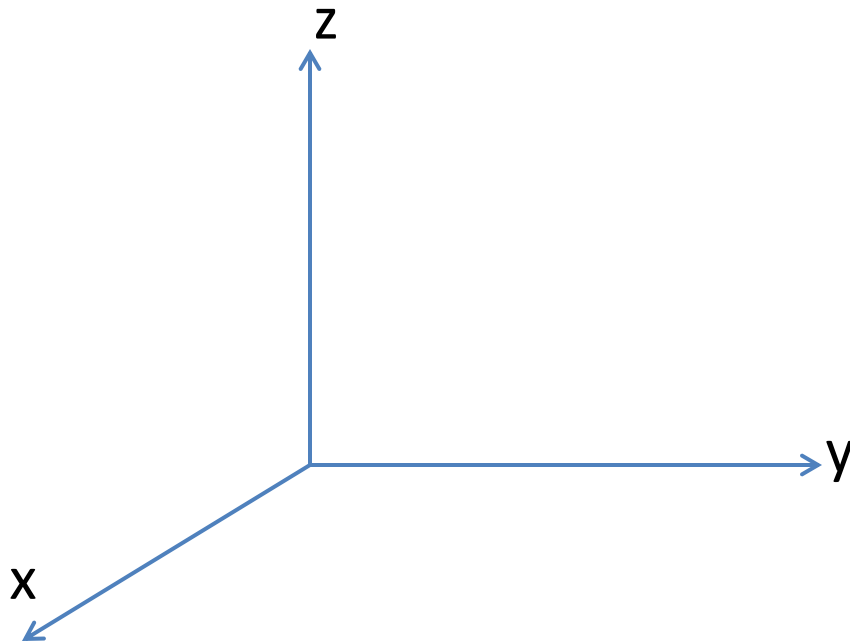
- The **second-order differential condition** for any arbitrary nonzero value of z :
 - For a maximum of z : $d^2z < 0$.
 - For a minimum of z : $d^2z > 0$.

TWO CHOICE VARIABLE OPTIMIZATION

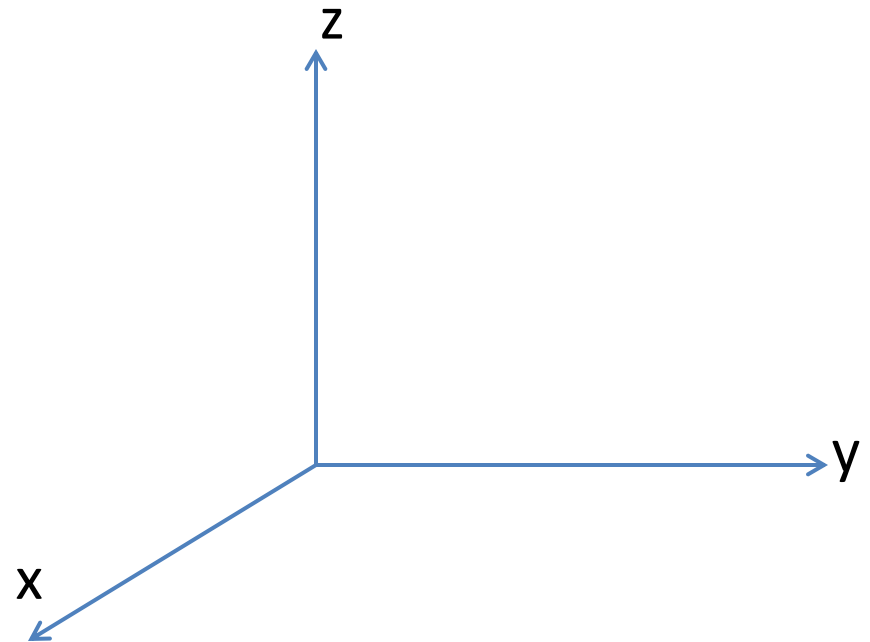
Extreme Values of a Function of Two Variables:

$$z = f(x, y)$$

Maximum value



Minimum value



First-Order Condition

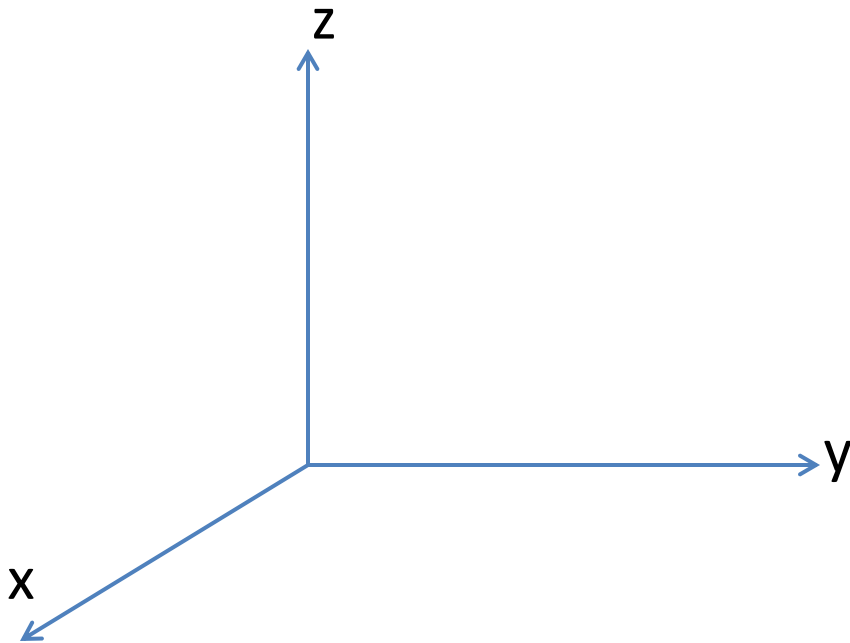
- For $z = f(x, y)$, the first-order necessary condition is:
 $dz = 0$ for arbitrary values of dx and dy , not both zero.
- The total differential is:
 $dz = f_x dx + f_y dy$
 $dz = 0 \rightarrow f_x = f_y = 0.$

Theorem: A differentiable function $z = f(x, y)$ can only have a maximum or minimum at an interior point (x_0, y_0) if it is a stationary point. That is, if the point $(x, y) = (x_0, y_0)$ satisfies the two F.O.C. equations:

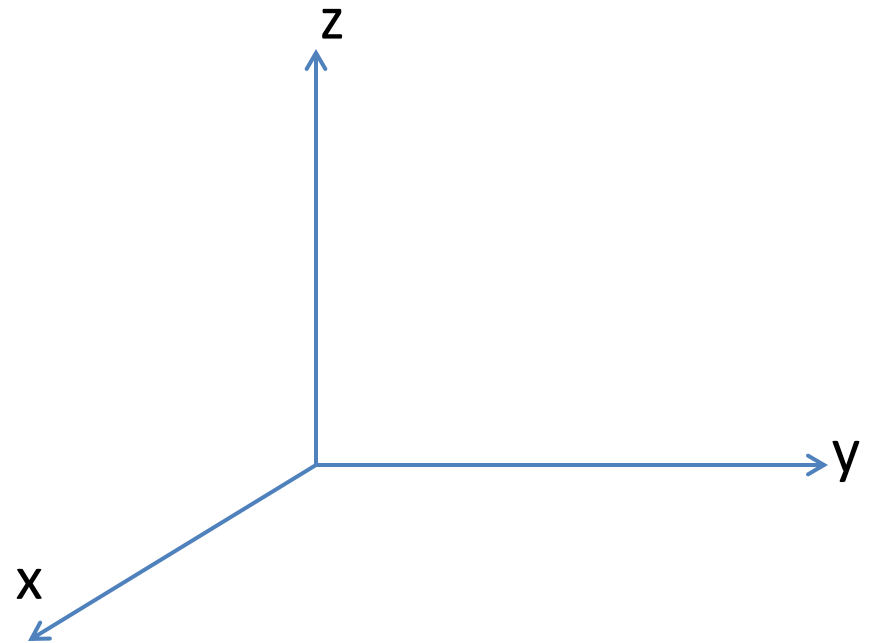
$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

The FOC is necessary but *not sufficient* condition.

Saddle Point



Inflection Point



Second-Order Condition

- Second-order partial derivatives:

$$\triangleright f_{xx} = \quad ; \quad f_{xy} =$$

$$\triangleright f_{yy} = \quad ; \quad f_{yx} =$$

- Second-order total differentials:

Second-Order Condition

- **Example:** Let $z = x^3 + 5xy - y^2$. Find dz and d^2z . Then, evaluate dz and d^2z at $x = 1$ and $y = 2$.

Second-Order Condition

- Given that the first-order necessary condition is satisfied, the **second-order sufficient condition** for $z = f(x, y)$ is:
 - For **maximum**, $d^2z < 0$ for arbitrary values of dx and dy , not both zero.
 - For **minimum**, $d^2z > 0$ for arbitrary values of dx and dy , not both zero.
- Equivalent conditions on second-order partial derivatives:
 1. $d^2z < 0$ iff $f_{xx} < 0$; $f_{yy} < 0$; and $f_{xx} f_{yy} > (f_{xy})^2$.
 2. $d^2z > 0$ iff $f_{xx} > 0$; $f_{yy} > 0$; and $f_{xx} f_{yy} > (f_{xy})^2$.

Note: The second-order partial derivatives are to be evaluated at the stationary point where $f_x = f_y = 0$.

Necessary and Sufficient Conditions for a Maximum and a Minimum

Condition	Maximum	Minimum
First-order necessary	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient	$f_{xx} f_{yy} < 0$ <i>and</i> $f_{xx} f_{yy} > (f_{xy})^2$	$f_{xx} f_{yy} > 0$ <i>and</i> $f_{xx} f_{yy} > (f_{xy})^2$

Examples: 2-choice variable optimization

- Example1: $z = f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$.

Second-Order Total Differential as a Quadratic Form

- Consider dx and dy as variables and partial derivatives as coefficients; i.e.,

$$u = dx \quad ; \quad v = dy$$

$$a = f_{xx} \quad ; \quad b = f_{yy} \quad ; \quad h = f_{xy} [=f_{yx}]$$

- From $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$, and let $q = d^2z$.

Rewrite d^2z as:

- Question: What restriction must be placed on a , b , and h , in order to determine a *definite* sign of q ?

Positive and Negative Definiteness

- A quadratic form is said to be

<p><i>positive definite</i></p> <p><i>positive semidefinite</i></p> <p><i>negative semidefinite</i></p> <p><i>negative definite</i></p>	}	if q is invariably	{	<p><i>positive (> 0)</i></p> <p><i>non-negative (≥ 0)</i></p> <p><i>non-positive (≤ 0)</i></p> <p><i>negative (< 0)</i></p>
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- If q changes signs when the variables (dx, dy) take different values, then q is said to be *indefinite* (e.g. saddle point).

Determinantal Test for Sign Definiteness

- From $q = au^2 + 2huv + bv^2$

➤ q is $\begin{cases} \text{positive definite} \\ \text{negative definite} \end{cases}$ iff $\begin{cases} a > 0 \\ a < 0 \end{cases}$ and $ab - h^2 > 0$.

Determinantal Test for Sign Definiteness

- Rewrite $q = au^2 + 2huv + bv^2$ in the matrix form:

➤ q is $\begin{cases} \text{positive definite} \\ \text{negative definite} \end{cases}$ iff $\begin{cases} |a| > 0 \\ |a| < 0 \end{cases}$ and $\begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$.

where

$|a| = a$ is the first leading principal minor of D ;

$\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ is the second leading principal minor of D .

Determinantal Test for Sign Definiteness

- Thus, the second-order total differential

1. d^2z is *positive definite* iff $f_{xx} > 0$ & $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$

→ z is a *minimum*.

2. d^2z is *negative definite* iff $f_{xx} < 0$ & $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$

→ z is a *maximum*.

- In the matrix form $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$,

The determinant with the second-order partial derivatives as its elements is called a “**Hessian determinant**”:

Example: Multiproduct Firm (1)

- Competitive firm

Let $p_1 = 6$, $p_2 = 9$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

Find Q_1 and Q_2 that maximize the firm's profit.

Example: Multiproduct Firm (2)

- Monopolist

Let $p_1 = 35 - Q_1$, $p_2 = 33 - Q_2$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

Find Q_1 and Q_2 that maximize the firm's profit.

Example: Multiplant Firm

- Let $P = 25$, $TC_1 = 2Q_1^2 + 5Q_1 + 10$, and $TC_2 = 2Q_2^2 + 3Q_2 + 15$.
Find Q_1 and Q_2 that maximize the firm's profit.

Example:

Multimarket Monopoly (Price Discrimination)

- Let $R = R_1(Q_1) + R_2(Q_2)$ and $C = C(Q)$ where $Q = Q_1 + Q_2$. Find the FONC and SOSC for profit maximization.

Example: Price Discrimination

- Let $p_1 = 22 - 2Q_1$, $p_2 = 10 - 0.5Q_2$ and $TC = 2Q + 5$.
Find Q_1 , Q_2 , p_1 , and p_2 that maximize the firm's profit.
- Max $\pi = (22 - 2Q_1)Q_1 + (10 - 0.5Q_2)Q_2 - [2(Q_1+Q_2)+ 5]$
- FONC: $\pi_1 = 22 - 4Q_1 - 2 = 0$
 $\pi_2 = 10 - Q_2 - 2 = 0$
 ➔ $Q_1^* = 5, Q_2^* = 8$
- SOSOC: $\pi_{11} = -4, \pi_{22} = -1, \pi_{12} = 0$
 $|H_1| < 0$ & $|H| = [(4)(1)-0] > 0$
 ➔ SOSOC for max are satisfied.

Example: Input Decision of a firm

- Let $Q = f(K, L) = 5K^{0.5}L^{0.25}$, $P = 4$, $w = 10$, $r = 5$. Find K^* and L^* that maximize profit.

MULTIVARIABLE OPTIMIZATION

Objective Functions with 3 Choice Variables

- Let $z = f(x_1, x_2, x_3)$. $\rightarrow dz =$
- **First-order necessary condition** for an extremum of z is:
- Second-order total differential:
 $d^2z =$
- ➔ Hessian matrix:

Objective Functions with 3 Choice Variables

- The leading principal minors of the Hessian matrix are:
- Thus, **second-order sufficient condition** for an extremum of z :
 1. z^* is a *minimum* if $|H_1| > 0$; $|H_2| > 0$; $|H_3| > 0$
(i.e. d^2z is *positive definite*.)
 2. z^* is a *maximum* if $|H_1| < 0$; $|H_2| > 0$; $|H_3| < 0$
(i.e. d^2z is *negative definite*.)

Note: All the leading principal minors are evaluated at the stationary points where $f_1 = f_2 = f_3 = 0$.

n -variable Optimization

- Let $z = f(x_1, x_2, \dots, x_n)$.
- Total differential:
- First-order necessary condition:

- Second-order sufficient condition: Consider the Hessian matrix:
$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

where $|H_1|, |H_2|, \dots, |H_n|$ are the leading principal minors.

For a *mimimum* in z , all the principal minors must be *positive*.

For a *maximum* in z , all the principal minors *alternate is sign, the first one being negative*.

Necessary and Sufficient Conditions for a Maximum and Minimum (n - choice variables)

Condition	Maximum	Minimum
First-order necessary	$f_1 = f_2 = \dots = f_n = 0$	$f_1 = f_2 = \dots = f_n = 0$
Second-order sufficient	$ H_1 < 0, H_2 > 0,$ $ H_3 < 0, \dots$ or $(-1)^i H_i > 0$ for $i = 1, 2, \dots, n$	$ H_1 > 0, H_2 > 0,$ $ H_3 > 0, \dots$ Or $ H_i > 0$ for $i = 1, 2, \dots, n$

Example: Multimarket Monopoly

- Let $p_1 = 63 - 4Q_1$, $p_2 = 105 - 5Q_2$, $p_3 = 75 - 6Q_2$ and $TC = 15Q + 20$. Find Q_i and p_i , for $i = 1, 2, 3$, that maximize the firm's profit.