CHAPTER 4

Basic Matrix Algebra and Applications

Topics: Application of Matrix Algebra

Outline:

- How to use matrix to solve system of simultaneous equations
- Partial market equilibrium
- The effect of specific tax on market equilibrium
- Simple Macroeconomic Model
- IS – LM

How to use matrix to solve system of simultaneous equations

When we have the system of simultaneous equations:

1.) Identify endogenous variables, exogenous variables, parameters

2.) Reduce the system of equations to have number of equations equal to number of endogenous variables and to have only necessary information

3.) Rearrange every equations in the system so that all endogenous variables are on one side.

From:

$$ax = k - by$$
$$dy = h - cx$$

To:

$$ax + by = k$$
$$cx + dy = h$$

4.) Rewrite the system into matrix form:

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} k \\ h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ h \end{bmatrix}$$

$$\boxed{A_{n\times n} x_{n\times 1} = d_{n\times 1}}$$

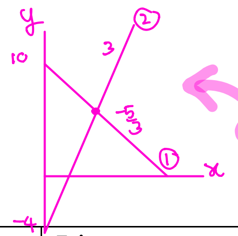
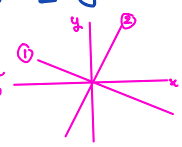
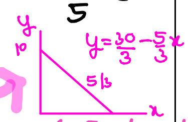
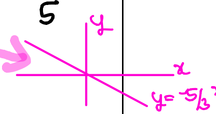
A is matrix of parameters that are in front of endogenous variables.

x is column vector of endogenous variables to solve for its final values as a function of exogenous variables and parameters

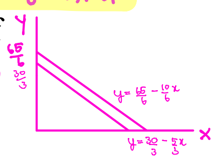
d is column vector of parameters that don't multiply with endogenous variables and exogenous variables

5.) Check det A: note det A = 0 when $\begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = kab - kab = 0$
$kr_2 = r_1$

To find solution to linear system of equations: $A_{n\times n}x_{n\times 1} = d_{n\times 1}$

Det A \ Vector d		$d_{n\times1} \neq 0$	$d_{n\times1} = 0$ (homogenous equation system)
$\lvert A\rvert \neq 0$ (Linear independent eq) Matrix A is non-singular A^{-1} *exist* $\begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -10 - 18 = -28 \neq 0$ $5x + 3y = 30 \Rightarrow y = 10 - \frac{5}{3}x$ $6x - 2y = 8 \Rightarrow y = -4 + 3x$ 		Unique solution $x = A^{-1}d \neq \underline{0}$ e.g. $\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 8 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 38 \\ 8 \end{bmatrix}$ cross lines → one solution 2 independent eqs	Unique solution $x = A^{-1}\underline{0} = \underline{0}$ e.g. $\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ① $5x + 3y = 0$, $y = -\frac{5}{3}x$ ② $6x - 2y = 0$, $y = 3x$  2 endo var
$\lvert A\rvert = 0$ Matrix A singular A^{-1} *does not exist*	Linear dependent eq. $\lvert A_k\rvert = 0, \lvert A\rvert = 0$	Infinite solutions (not including $x = \underline{0}$) e.g. $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$ $5x + 3y = 30$ $x = \frac{30 - 3y}{5}$ $x = \frac{0}{0}$ $y = \frac{0}{0}$  same line, infinite solution	Infinite solutions (including $x = \underline{0}$) e.g. $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $5x + 3y = 0$ $x = \frac{-3y}{5}$ $x = 0/0$ $y = 0/0$  1 eq 2 endo var. not enough info
	Inconsistent eq. $\lvert A_k\rvert \neq 0, \lvert A\rvert = 0$	No solution $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 65 \end{bmatrix}$	n/a

$5x + 3y = 30 \rightarrow y = \frac{30}{3} - \frac{5}{3}x$
$10x + 6y = 65 \rightarrow y = \frac{65}{6} - \frac{10}{6}x$
parallel lines, no solution



6.) Use inverse matrix or Cramer's rule to find the solution of the system of simultaneous equations:

$$x = A^{-1}d \quad or \quad x_k = \frac{\lvert A_k\rvert}{\lvert A\rvert}$$

Note: If determinant is found by using Cofactor for elements from the WRONG row or column, the determinant will be zero!!!!!

$$\begin{vmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \qquad |C_{21}| = -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \qquad |C_{22}| = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 \qquad |C_{23}| = -\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$|A| = 4(-3) + 1(10) + 2(1) = 0$$

Exercise: Consider the following system of equations and examine whether there is(are) solution(s) to the system or not.

1.)
$$\begin{aligned} 4x + y + 2z &= 5 \\ 5x + 2y + z &= 7 \\ x + z &= 1 \end{aligned}$$

2.)
$$\begin{aligned} 3x &= 12 - 5y + 4z \\ 6x &= 9 + 10y \\ 8x &= 10 + 7y - 3z \end{aligned}$$

Partial market equilibrium

rearrange

Demand: $Q^D = a - bP$ $\qquad Q^D + 0Q^S + bP = a$

Supply: $Q^S = -c + dP \Rightarrow \quad 0Q^D + Q^S - dP = -c$

Equilibrium: $Q^D = Q^S$ $\qquad Q^D - Q^S + 0P = 0$

endo Q^D, Q^S, P

turn into matrix form

$$A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \qquad x = \begin{bmatrix} Q^D \\ Q^S \\ P \end{bmatrix} \qquad D = \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$$

$$Q^D = \frac{\begin{vmatrix} a & 0 & b \\ -c & 1 & -d \\ 0 & -1 & 0 \end{vmatrix}}{-b-d} = \frac{bc - ad}{-b-d} = \frac{ad - bc}{b+d}$$

$$Q^S = \frac{\begin{vmatrix} 1 & a & b \\ 0 & -c & -d \\ 1 & c & 0 \end{vmatrix}}{-b-d} = \frac{-ad+bc}{-b-d} = \frac{ad-bc}{b+d}$$

$$P = \frac{\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & -c \\ 1 & -1 & 0 \end{vmatrix}}{-b-d} = \frac{-a-c}{-b-d} = \frac{a+c}{b+d}$$

Note: 1.) If we assume $Q^D = Q^S = Q$

$$Q = a - bP$$
$$Q = -c + dP$$

rearrange

$$Q + bP = a$$
$$Q - dP = -c$$

turn into matrix form

$$\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} a \\ -c \end{bmatrix}$$

$|A| = -d-b \neq 0$ for A^{-1} to exist

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \dfrac{\begin{vmatrix} a & b \\ -c & -d \end{vmatrix}}{-d-b} \\[2ex] \dfrac{\begin{vmatrix} 1 & a \\ 1 & -c \end{vmatrix}}{-d-b} \end{bmatrix} = \begin{bmatrix} \dfrac{-ad+bc}{-d-b} \\[2ex] \dfrac{-c-a}{-d-b} \end{bmatrix} = \begin{bmatrix} \dfrac{ab-bc}{b+d} \\[2ex] \dfrac{a+c}{b+d} \end{bmatrix}$$

2.) If we collect specific tax from supplier t baht per unit

$Q^d = Q^s$

$Q^d = a - bP^d$

$Q^s = -c + dP^s$

$P^s = P^d - t$

let $P^d = P$

with $Q^d = Q^s = Q$

make it as simple as it can be

$Q = a - bP$

$Q = -c + dP - dt$

rearrange

$Q + bP = a$

$Q - dP = -c - dt$

turn into matrix form

$$\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} a \\ -c-dt \end{bmatrix}$$

Solve for Q, P by A^{-1} or Cramer's rule

$$Q^* = \frac{\begin{vmatrix} a & b \\ -c-dt & -d \end{vmatrix}}{\begin{vmatrix} 1 & b \\ 1 & -d \end{vmatrix}} = \frac{-ad - b(-c-dt)}{-d-b} = \frac{ad - bc - bdt}{b+d} \quad \#$$

$$P^* = \frac{\begin{vmatrix} 1 & a \\ 1 & -c-dt \end{vmatrix}}{\begin{vmatrix} 1 & b \\ 1 & -d \end{vmatrix}} = \frac{-a - c - dt}{-d-b} = \frac{a + c + dt}{b+d} = P^*_{AT} = P^*_d \quad \#$$

$$P^*_s = P^*_d - t = \frac{a + c - bt}{b+d} \quad \#$$

Exercise:

1.)

$Q^D = 24 - 4P$

$Q^S = 13P - 27 \qquad (P^* = 3, Q^* = 12)$

$Q^D = Q^S$

2.) Consider a simple model in which two commodities are related to each other

Good 1	Good 2
$Q_{d1} = Q_{s1}$	$Q_{d2} = Q_{s2}$
$Q_{d1} = a_0 + a_1P_1 + a_2P_2$	$Q_{d2} = c_0 + c_1P_1 + c_2P_2$
$Q_{s1} = b_0 + b_1P_1 + b_2P_2$	$Q_{s2} = d_0 + d_1P_1 + d_2P_2$

Find the equilibrium prices and quantities in good 1 and good 2 $\Rightarrow$ 4x4 with endo Q_1, Q_2, P_1, P_2

Simple Macroeconomic Model

Consider simple Keynesian crossing model:

$$\left.\begin{aligned}
&(1.)\ Y = C + I + G \\
&(2.)\ C = a + bY_d \\
&(3.)\ Y_d = Y - T \\
&(4.)\ I = I_0 \\
&(5.)\ G = G_0
\end{aligned}\right\}\ \text{substitute out some equations}$$

endo : Y, C, Y_d

$Y = C + I_0 + G_0$

$C = a + bY_d$

$Y_d = Y - T$

rearrange

$Y - C + 0Y^d = I_0 + G_0$

$0Y + C - bY^d = a$

$-Y + 0C + Y^d = -T$

turn into matrix form

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -b \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ Y^d \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \\ -T \end{bmatrix}$$

coeffi in front of Y

of C

of Y^d

$|A| = 1-b \neq 0$ since $0<b<1$ $\therefore$ A^{-1} exist

By Cramer's rule

$$Y^* = \frac{|A_1|}{|A|}$$

$$Y^* = \frac{\begin{vmatrix} I_0+G_0 & -1 & 0 \\ a & 1 & -b \\ -T & 0 & 1 \end{vmatrix}}{1-b} = \frac{I_0+G_0-bT+a}{1-b} = \frac{a-bT+I_0+G_0}{1-b} \quad \#$$

$$C^* = \frac{|A_2|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & I_0+G_0 & 0 \\ 0 & a & -b \\ -1 & -T & 1 \end{vmatrix}}{1-b} = \frac{a+b(I_0+G_0)-bT}{1-b} \quad \#$$

$$Y^{d*} = \frac{|A_3|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & -1 & I_0+G_0 \\ 0 & 1 & a \\ -1 & 0 & -T \end{vmatrix}}{1-b} = \frac{-T+a+I_0+G_0}{1-b} = \frac{a-T+I_0+G_0}{1-b}$$

$$\left.\begin{aligned} Y &= C+I_0+G_0 \\ C &= a+bY_d \\ Y_d &= Y-T \end{aligned}\right\} \xRightarrow{\text{substitute}} \begin{aligned} Y &= C+I_0+G_0 \\ C &= a+bY-bT \end{aligned}$$

rearrange

$$\begin{aligned} Y - C &= I_0+G_0 \\ -bY + C &= a-bT \end{aligned}$$

$$\underset{ad-bc}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \neq \underset{bc-ad}{\begin{vmatrix} c & d \\ a & b \end{vmatrix}}$$

Turn into matrix

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a - bT \end{bmatrix}$$

$|A| = 1-b \neq 0$ A^{-1} exists

$$\begin{bmatrix} Y \\ C \end{bmatrix} = \frac{1}{1-b} \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} I_0 + G_0 \\ a - bT \end{bmatrix}$$

$Y^* = \dfrac{a - bT + I_0 + G_0}{1-b}$, $C^* = \dfrac{a - bT + b(I_0 + G_0)}{1-b}$

If Gov. increase G and T equally, $\Delta Y^* = \dfrac{-b\Delta T}{1-b} + \dfrac{\Delta G}{1-b} = 1$; $\Delta T = \Delta G$

(Balanced-budget multiplier = a measure of the change in Y caused by equal changes in government purchases and taxes)

IS – LM

The Good Market:

$Y = C + I + G$

$C = C_0 + b(1-t)Y$

$I = I_0 - er$

$G = G_0$

The Money Market:

$M_d = fY - \beta r$

$M_s = M_0$

$M_d = M_s$

endo Y, C, I, T, M_d, r

$$\left.\begin{aligned} Y &= C + I + G_0 \\ C &= C_0 + b(1-t)Y \\ I &= I_0 - er \\ M_0 &= fY - \beta r \end{aligned}\right\} \text{4 endo , 4 equations}$$

rearrange

$$
\begin{aligned}
Y - C - I + 0r &= G_0 \\
-b(1-t)Y + C + 0I + 0r &= C_0 \\
0Y + 0C + I + er &= I_0 \\
fY + 0C + 0I - \beta r &= M_0
\end{aligned}
$$

Turn into matrix form

$$
\begin{bmatrix} 1 & -1 & -1 & 0 \\ -b(1-t) & 1 & 0 & 0 \\ 0 & 0 & 1 & e \\ f & 0 & 0 & -\beta \end{bmatrix}
\begin{bmatrix} Y \\ C \\ I \\ r \end{bmatrix}
=
\begin{bmatrix} G_0 \\ C_0 \\ I_0 \\ M_0 \end{bmatrix}
$$

find det A : laplace expansion by row 3

$$|A| = 1(-1)^{3+3} M_{33} + e(-1)^{3+4} M_{34}$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -b(1-t) & 1 & 0 \\ f & 0 & -\beta \end{vmatrix} - e \begin{vmatrix} 1 & -1 & -1 \\ -b(1-t) & 1 & 0 \\ f & 0 & 0 \end{vmatrix}$$

$$= -\beta + \beta b(1-t) - e(f)$$

$$= -[ef + \beta(1 - b(1-t))] \neq 0 \text{ if}$$

$$ef + \beta \neq \beta b(1-t)$$

Cramer's rule

$$Y^* = \frac{|A_1|}{|A|}$$

$$= \frac{\begin{vmatrix} G_0 & -1 & -1 & 0 \\ C_0 & 1 & 0 & 0 \\ I_0 & 0 & 1 & e \\ M_0 & 0 & 0 & -\beta \end{vmatrix}}{|A|} = \frac{(-1)C_{12} + (1)C_{22}}{|A|}$$

$$= \frac{(-1)(-1)^{1+2}\begin{vmatrix} C_0 & 0 & 0 \\ I_0 & 1 & e \\ M_0 & 0 & -\beta \end{vmatrix} + (1)(-1)^{2+2}\begin{vmatrix} G_0 & -1 & 0 \\ I_0 & 1 & e \\ M_0 & 0 & -\beta \end{vmatrix}}{|A|}$$

$$Y^* = \frac{-C_0\beta + (-G_0\beta - eM_0 - \beta I_0)}{-[ef + \beta(1-b(1-t))]}$$

$$= \frac{C_0\beta + G_0\beta + eM_0 + \beta I_0}{ef + \beta(1-b(1-t))}$$

$$= \frac{\beta(C_0 + G_0 + I_0) + eM_0}{ef + \beta(1-b(1-t))}$$

$$r^* = \frac{|A_4|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & -1 & -1 & G_0 \\ -b(1-t) & 1 & 0 & C_0 \\ 0 & 0 & 1 & I_0 \\ f & 0 & 0 & M_0 \end{vmatrix}}{|A|}$$

$$= \frac{(-1)C_{12} + (1)C_{22}}{|A|}$$

$$= \frac{\begin{vmatrix} -b(1-t) & 0 & C_0 \\ 0 & 1 & I_0 \\ f & 0 & M_0 \end{vmatrix} + \begin{vmatrix} 1 & -1 & G_0 \\ 0 & 1 & I_0 \\ f & 0 & M_0 \end{vmatrix}}{|A|}$$

$$= \frac{-b(1-t)M_0 - C_0 f + M_0 - I_0 f - G_0 f}{-[ef + \beta(1-b(1-t))]}$$

$$= \frac{(C_0 + I_0 + G_0)f - (1 - b(1-t))M_0}{ef + \beta(1-b(1-t))}$$

$$I^* = \frac{|A_3|}{|A|}$$

$$C^* = \frac{|A_2|}{|A|}$$

Or use matrix algebra to solve IS-LM equations

$$\left.\begin{aligned} Y &= C + I + G_0 \\ C &= C_0 + b(1-t)Y \\ I &= I_0 - er \\ M_0 &= fY - \beta r \end{aligned}\right\} \begin{aligned} &Y = C_0 + b(1-t)Y + I_0 - er + G_0 \\ \Rightarrow\ &(1-b(1-t))Y + er = C_0 + I_0 + G_0 \quad \text{IS} \\ &fY - \beta r = M_0 \quad \text{LM} \end{aligned}$$

⇒ Transfer to matrix form

$$\begin{bmatrix} (1-b(1-t)) & e \\ f & -\beta \end{bmatrix} \begin{bmatrix} Y \\ r \end{bmatrix} = \begin{bmatrix} C_0+I_0+G_0 \\ M_0 \end{bmatrix}$$

$$|A| = -\beta(1-b(1-t)) - ef \neq 0 \quad \text{if} \quad \beta(1-b(1-t)) \neq -ef$$

$$Y^* = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} C_0+I_0+G_0 & e \\ M_0 & -\beta \end{vmatrix}}{|A|}$$

$$= \frac{-\beta(C_0+I_0+G_0) - eM_0}{-\beta(1-b(1-t)) - ef}$$

$$= \frac{\beta(C_0+I_0+G_0) + eM_0}{\beta(1-b(1-t)) + ef} \quad \#$$

$$r^* = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1-b(1-t) & C_0+I_0+G_0 \\ f & M_0 \end{vmatrix}}{-\beta(1-b(1-t)) - ef}$$

$$= \frac{1-b(1-t)M_0 - f(C_0+I_0+G_0)}{-\beta(1-b(1-t)) - ef}$$

$$= \frac{f(C_0+I_0+G_0) - (1-b(1-t))M_0}{ef + \beta(1-b(1-t))}$$

$I^* = I_0 - er^*$

$C^* = C_0 + b(1-t)Y^*$

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