Fixed income instrument

EE431

Semester 1/2017

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Topics

- Bond pricing and yield to maturity
- Bond risks and measuring price risk.
- Bond price and interest rate determination
- Risk and term structure of interest rates

Reading lists

- Required text:

  Mishkin Ch 4 – Ch 6 (required!)

  Fabozzi Ch 2 - **Ch** 5 (optional, a bit more finance)

This week

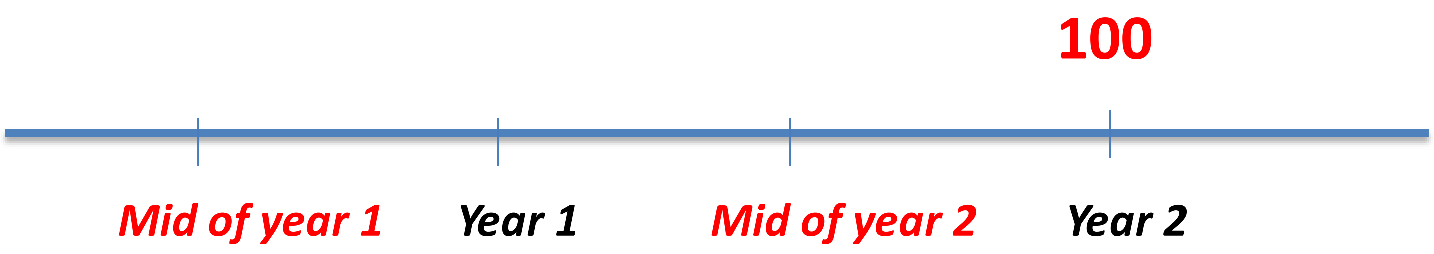
- Bond valuation and Yield to maturity
- Rate of return on bond investment

Valuation of credit asset

- Credit contract: with the following details pre-specified in advance
  - Cash flow payment
  - Date of payment
  - Time to maturity
- Tradable v.s. Non-tradable credit asset
  - Bank loan (non-standardized credit asset)
  - Bond (standardized credit asset)

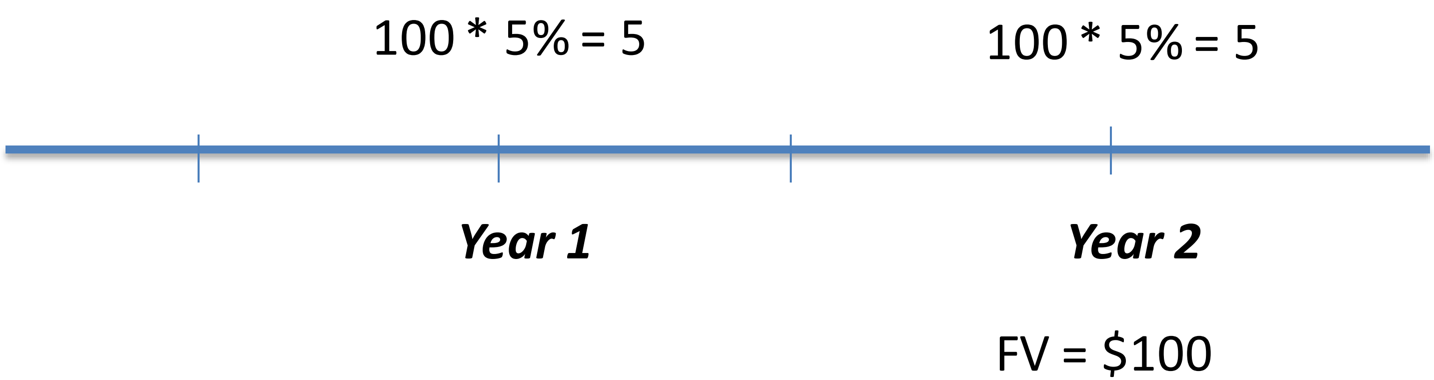
Tradable credit asset: zero-coupon bond

- Government Discount bond, with the face value of 100, repay in 2 year from now.



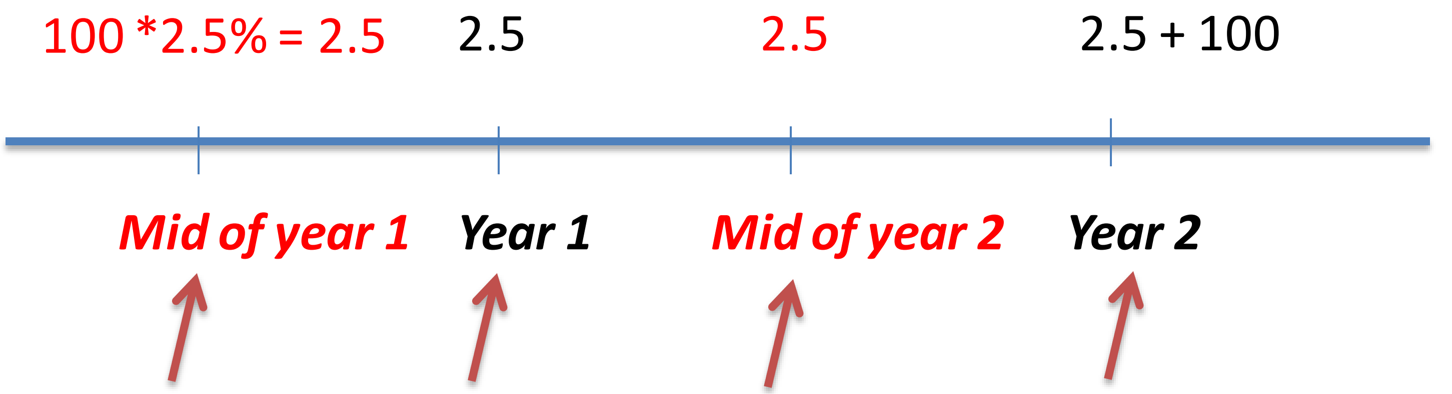
Tradable credit asset: coupon bond 1

- A 5% 2-year coupon bond with its face value of $100.



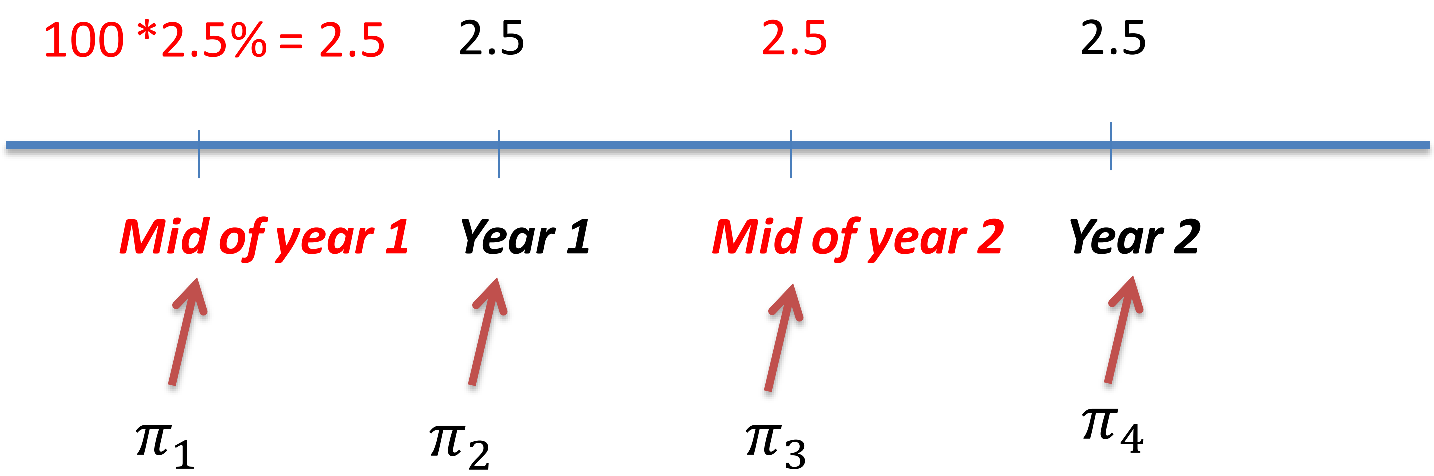
Tradable credit asset: coupon bond 3

- A 2-year coupon bond with semi-annual payment has 5% coupon rate and the face value of $100



Tradable credit asset: coupon bond 3

- A 2-year coupon bond with semi-annual payment has 5% coupon rate and the face value of $100
- Inflation-indexed bond



$$Cashflow\ at\ the\ Maturity\ date\ = FV * (1+\pi_1)(1+\pi_2)\,(1+\pi_3)\,(1+\pi_4)$$

Understanding the nature of bond pricing

- Principle of asset valuation

- Bond pricing formula

$$P_t = V_t = \frac{CF_1}{1 + r_T} + \frac{CF_2}{(1 + r_T)^2} + \cdots + \frac{CF_T}{(1 + r_T)^T}$$

$r_T = discount\ rate$

Example 1:
Discount bond with FV = 100

- Cash flow is 100 บาท
- T = 2.
- If the market discount rate is 5% p.a., the value of cash flow, measured in terms of present value is

Fair value = price

$$\frac{100}{(1+\mathbf{0.05})^2} = \$90.70$$

Example 1

- Look at the present value factor table
  - Present value factor (PVF) shows the present value of $1 obtained t-period from now with the discount rate r% per period.

Present value interest factor of $1 per period at r% for t periods, PVIF(r,t).

Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%	16%	17%	18%	19%	20%
1	0.990	0.980	0.971	0.962	0.952	0.943	0.935	0.926	0.917	0.909	0.901	0.893	0.885	0.877	0.870	0.862	0.855	0.847	0.840	0.833
2	0.980	0.961	0.943	0.925	**0.907**	0.890	0.873	0.857	0.842	0.826	0.812	0.797	0.783	0.769	0.756	0.743	0.731	0.718	0.706	0.694
3	0.971	0.942	0.915	0.889	0.864	0.840	0.816	0.794	0.772	0.751	0.731	0.712	0.693	0.675	0.658	0.641	0.624	0.609	0.593	0.579
4	0.961	0.924	0.888	0.855	0.823	0.792	0.763	0.735	0.708	0.683	0.659	0.636	0.613	0.592	0.572	0.552	0.534	0.516	0.499	0.482
5	0.951	0.906	0.863	0.822	0.784	0.747	0.713	0.681	0.650	0.621	0.593	0.567	0.543	0.519	0.497	0.476	0.456	0.437	0.419	0.402
6	0.942	0.888	0.837	0.790	0.746	0.705	0.666	0.630	0.596	0.564	0.535	0.507	0.480	0.456	0.432	0.410	0.390	0.370	0.352	0.335
7	0.933	0.871	0.813	0.760	0.711	0.665	0.623	0.583	0.547	0.513	0.482	0.452	0.425	0.400	0.376	0.354	0.333	0.314	0.296	0.279
8	0.923	0.853	0.789	0.731	0.677	0.627	0.582	0.540	0.502	0.467	0.434	0.404	0.376	0.351	0.327	0.305	0.285	0.266	0.249	0.233
9	0.914	0.837	0.766	0.703	0.645	0.592	0.544	0.500	0.460	0.424	0.391	0.361	0.333	0.308	0.284	0.263	0.243	0.225	0.209	0.194
10	0.905	0.820	0.744	0.676	0.614	0.558	0.508	0.463	0.422	0.386	0.352	0.322	0.295	0.270	0.247	0.227	0.208	0.191	0.176	0.162
11	0.896	0.804	0.722	0.650	0.585	0.527	0.475	0.429	0.388	0.350	0.317	0.287	0.261	0.237	0.215	0.195	0.178	0.162	0.148	0.135
12	0.887	0.788	0.701	0.625	0.557	0.497	0.444	0.397	0.356	0.319	0.286	0.257	0.231	0.208	0.187	0.168	0.152	0.137	0.124	0.112
13	0.879	0.773	0.681	0.601	0.530	0.469	0.415	0.368	0.326	0.290	0.258	0.229	0.204	0.182	0.163	0.145	0.130	0.116	0.104	0.093
14	0.870	0.758	0.661	0.577	0.505	0.442	0.388	0.340	0.299	0.263	0.232	0.205	0.181	0.160	0.141	0.125	0.111	0.099	0.088	0.078
15	0.861	0.743	0.642	0.555	0.481	0.417	0.362	0.315	0.275	0.239	0.209	0.183	0.160	0.140	0.123	0.108	0.095	0.084	0.074	0.065
16	0.853	0.728	0.623	0.534	0.458	0.394	0.339	0.292	0.252	0.218	0.188	0.163	0.141	0.123	0.107	0.093	0.081	0.071	0.062	0.054
17	0.844	0.714	0.605	0.513	0.436	0.371	0.317	0.270	0.231	0.198	0.170	0.146	0.125	0.108	0.093	0.080	0.069	0.060	0.052	0.045
18	0.836	0.700	0.587	0.494	0.416	0.350	0.296	0.250	0.212	0.180	0.153	0.130	0.111	0.095	0.081	0.069	0.059	0.051	0.044	0.038
19	0.828	0.686	0.570	0.475	0.396	0.331	0.277	0.232	0.194	0.164	0.138	0.116	0.098	0.083	0.070	0.060	0.051	0.043	0.037	0.031
20	0.820	0.673	0.554	0.456	0.377	0.312	0.258	0.215	0.178	0.149	0.124	0.104	0.087	0.073	0.061	0.051	0.043	0.037	0.031	0.026
25	0.780	0.610	0.478	0.375	0.295	0.233	0.184	0.146	0.116	0.092	0.074	0.059	0.047	0.038	0.030	0.024	0.020	0.016	0.013	0.010
30	0.742	0.552	0.412	0.308	0.231	0.174	0.131	0.099	0.075	0.057	0.044	0.033	0.026	0.020	0.015	0.012	0.009	0.007	0.005	0.004
35	0.706	0.500	0.355	0.253	0.181	0.130	0.094	0.068	0.049	0.036	0.026	0.019	0.014	0.010	0.008	0.006	0.004	0.003	0.002	0.002
40	0.672	0.453	0.307	0.208	0.142	0.097	0.067	0.046	0.032	0.022	0.015	0.011	0.008	0.005	0.004	0.003	0.002	0.001	0.001	0.001
50	0.608	0.372	0.228	0.141	0.087	0.054	0.034	0.021	0.013	0.009	0.005	0.003	0.002	0.001	0.001	0.001	0.000	0.000	0.000	0.000

Example 1

- From the table:

$$\text{PVF}(5\%,2) = \frac{1}{(1+0.05)^2} = 0.9070$$

- Present value $= 100*0.9070 = 90.70$

Example 2

- A 10-year bond is assumed to repay $1000 everything quarter. If the discount rate is 8% p.a., calculate the fair price.

Cautions

- T = period before the bond becomes matured!

  $$T = n * year \quad ;$$

  n = # payment in a year / year = # year

- Discount rate is usually quoted in p.a. percentage. Your discount rate must be adjusted to match with the number payment stream in a year.

  $$r_T = \frac{r}{n}$$ → **Discount rate p.a.**

Example 2

- Cash flow = 1000 for every quarter.
- 8% → 2% p.q.
- T = 10 * 4 = 40 quarter (period)

$$V_0 = \frac{1000}{1+0.02} + \frac{1000}{(1+0.02)^2} + \cdots + \frac{1000}{(1+0.02)^{40}}$$

$$= 1000\left\{\frac{1}{1+0.02} + \frac{1}{(1+0.02)^2} + \cdots + \frac{1}{(1+0.02)^{40}}\right\}$$

Geometric Series summation

Example 2

- Geometric Series summation

$$S_T = a + a^2 + a^3 + \cdots + a^T$$

$$aS_T = a^2 + a^3 + a^4 + \cdots + a^{T+1}$$

$$(1 - a)S_T = a - a^{T+1}$$

$$S_T = \frac{a - a^{T+1}}{1 - a}$$

$$S_\infty = \lim_{T \to \infty} S_T = \frac{a}{1 - a}; \quad -1 < a < 1$$

Example 2

- Since $a = \frac{1}{1+r}$, we then obtain that

$$S_T = \frac{\left(\frac{1}{1+r}\right) - \left(\frac{1}{1+r}\right)^{T+1}}{1 - \left(\frac{1}{1+r}\right)} = \frac{\mathbf{1 - \left(\frac{1}{1+r}\right)^{T}}}{\boldsymbol{r}}$$

Present value of an annuity factor = PVIFA(r%,T)

Present value interest factor of an (ordinary) annuity of $1 per period at r% for T periods, PVIFA(r,T).

Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%	16%	17%	18%	19%	20%
1	0.990	0.980	0.971	0.962	0.952	0.943	0.935	0.926	0.917	0.909	0.901	0.893	0.885	0.877	0.870	0.862	0.855	0.847	0.840	0.833
2	1.970	1.942	1.913	1.886	1.859	1.833	1.808	1.783	1.759	1.736	1.713	1.690	1.668	1.647	1.626	1.605	1.585	1.566	1.547	1.528
3	2.941	2.884	2.829	2.775	2.723	2.673	2.624	2.577	2.531	2.487	2.444	2.402	2.361	2.322	2.283	2.246	2.210	2.174	2.140	2.106
4	3.902	3.808	3.717	3.630	3.546	3.465	3.387	3.312	3.240	3.170	3.102	3.037	2.974	2.914	2.855	2.798	2.743	2.690	2.639	2.589
5	4.853	4.713	4.580	4.452	4.329	4.212	4.100	3.993	3.890	3.791	3.696	3.605	3.517	3.433	3.352	3.274	3.199	3.127	3.058	2.991
6	5.795	5.601	5.417	5.242	5.076	4.917	4.767	4.623	4.486	4.355	4.231	4.111	3.998	3.889	3.784	3.685	3.589	3.498	3.410	3.326
7	6.728	6.472	6.230	6.002	5.786	5.582	5.389	5.206	5.033	4.868	4.712	4.564	4.423	4.288	4.160	4.039	3.922	3.812	3.706	3.605
8	7.652	7.325	7.020	6.733	6.463	6.210	5.971	5.747	5.535	5.335	5.146	4.968	4.799	4.639	4.487	4.344	4.207	4.078	3.954	3.837
9	8.566	8.162	7.786	7.435	7.108	6.802	6.515	6.247	5.995	5.759	5.537	5.328	5.132	4.946	4.772	4.607	4.451	4.303	4.163	4.031
10	9.471	8.983	8.530	8.111	7.722	7.360	7.024	6.710	6.418	6.145	5.889	5.650	5.426	5.216	5.019	4.833	4.659	4.494	4.339	4.192
11	10.368	9.787	9.253	8.760	8.306	7.887	7.499	7.139	6.805	6.495	6.207	5.938	5.687	5.453	5.234	5.029	4.836	4.656	4.486	4.327
12	11.255	10.575	9.954	9.385	8.863	8.384	7.943	7.536	7.161	6.814	6.492	6.194	5.918	5.660	5.421	5.197	4.988	4.793	4.611	4.439
13	12.134	11.348	10.635	9.986	9.394	8.853	8.358	7.904	7.487	7.103	6.750	6.424	6.122	5.842	5.583	5.342	5.118	4.910	4.715	4.533
14	13.004	12.106	11.296	10.563	9.899	9.295	8.745	8.244	7.786	7.367	6.982	6.628	6.302	6.002	5.724	5.468	5.229	5.008	4.802	4.611
15	13.865	12.849	11.938	11.118	10.380	9.712	9.108	8.559	8.061	7.606	7.191	6.811	6.462	6.142	5.847	5.575	5.324	5.092	4.876	4.675
16	14.718	13.578	12.561	11.652	10.838	10.106	9.447	8.851	8.313	7.824	7.379	6.974	6.604	6.265	5.954	5.668	5.405	5.162	4.938	4.730
17	15.562	14.292	13.166	12.166	11.274	10.477	9.763	9.122	8.544	8.022	7.549	7.120	6.729	6.373	6.047	5.749	5.475	5.222	4.990	4.775
18	16.398	14.992	13.754	12.659	11.690	10.828	10.059	9.372	8.756	8.201	7.702	7.250	6.840	6.467	6.128	5.818	5.534	5.273	5.033	4.812
19	17.226	15.678	14.324	13.134	12.085	11.158	10.336	9.604	8.950	8.365	7.839	7.366	6.938	6.550	6.198	5.877	5.584	5.316	5.070	4.843
20	18.046	16.351	14.877	13.590	12.462	11.470	10.594	9.818	9.129	8.514	7.963	7.469	7.025	6.623	6.259	5.929	5.628	5.353	5.101	4.870
25	22.023	19.523	17.413	15.622	14.094	12.783	11.654	10.675	9.823	9.077	8.422	7.843	7.330	6.873	6.464	6.097	5.766	5.467	5.195	4.948
30	25.808	22.396	19.600	17.292	15.372	13.765	12.409	11.258	10.274	9.427	8.694	8.055	7.496	7.003	6.566	6.177	5.829	5.517	5.235	4.979
35	29.409	24.999	21.487	18.665	16.374	14.498	12.948	11.655	10.567	9.644	8.855	8.176	7.586	7.070	6.617	6.215	5.858	5.539	5.251	4.992
40	32.835	**27.355**	23.115	19.793	17.159	15.046	13.332	11.925	10.757	9.779	8.951	8.244	7.634	7.105	6.642	6.233	5.871	5.548	5.258	4.997
50	39.196	31.424	25.730	21.482	18.256	15.762	13.801	12.233	10.962	9.915	9.042	8.304	7.675	7.133	6.661	6.246	5.880	5.554	5.262	4.999

Example 2

- Cash flow = 1000 for every quarter.
- 8% → 2% p.q.
- T = 10 * 4 = 40 quarter (period)

$$V_0 = \frac{1000}{1+0.02} + \frac{1000}{(1+0.02)^2} + \cdots + \frac{1000}{(1+0.02)^{40}}$$

$$= 1000\left\{\frac{1}{1+0.02} + \frac{1}{(1+0.02)^2} + \cdots + \frac{1}{(1+0.02)^{40}}\right\}$$

$$= 1000\left\{\frac{\left(\frac{1}{1+0.02}\right) - \left(\frac{1}{1+0.02}\right)^{40+1}}{1-\left(\frac{1}{1+0.02}\right)}\right\} = 1000 * 27.355 = 27{,}355.$$

What do we learn from the example?

$$\text{Fair value/price} = \mathrm{CF} * \frac{1-\left(\frac{1}{1+r}\right)^{T}}{r} = \mathrm{CF} * \mathrm{PVIFA}(\mathrm{r}, \mathrm{T})$$

1. If known discount rate (r) ➔ implied market price.

2. If known market price

   ➔ Market-consistent discount rate.

   ➔ yield to maturity = interest rate on bond

Example 3: Implied discount rate

- If the market price of a semi-annual payment bond is $12,462, calculate the interest rate of this bond when the bond will be matured in 10 years from now.

Example 3 cont

- T = 20
- $12{,}462 = 1000\left\{\frac{1-\left(\frac{1}{1+r}\right)^{20}}{r}\right\}$
- $\left\{\frac{1-\left(\frac{1}{1+r}\right)^{20}}{r}\right\} = 12.462$

Present value interest factor of an (ordinary) annuity of $1 per period at r% for T periods, PVIFA(r,T).

Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%	16%	17%	18%	19%	20%
1	0.990	0.980	0.971	0.962	0.952	0.943	0.935	0.926	0.917	0.909	0.901	0.893	0.885	0.877	0.870	0.862	0.855	0.847	0.840	0.833
2	1.970	1.942	1.913	1.886	1.859	1.833	1.808	1.783	1.759	1.736	1.713	1.690	1.668	1.647	1.626	1.605	1.585	1.566	1.547	1.528
3	2.941	2.884	2.829	2.775	2.723	2.673	2.624	2.577	2.531	2.487	2.444	2.402	2.361	2.322	2.283	2.246	2.210	2.174	2.140	2.106
4	3.902	3.808	3.717	3.630	3.546	3.465	3.387	3.312	3.240	3.170	3.102	3.037	2.974	2.914	2.855	2.798	2.743	2.690	2.639	2.589
5	4.853	4.713	4.580	4.452	4.329	4.212	4.100	3.993	3.890	3.791	3.696	3.605	3.517	3.433	3.352	3.274	3.199	3.127	3.058	2.991
6	5.795	5.601	5.417	5.242	5.076	4.917	4.767	4.623	4.486	4.355	4.231	4.111	3.998	3.889	3.784	3.685	3.589	3.498	3.410	3.326
7	6.728	6.472	6.230	6.002	5.786	5.582	5.389	5.206	5.033	4.868	4.712	4.564	4.423	4.288	4.160	4.039	3.922	3.812	3.706	3.605
8	7.652	7.325	7.020	6.733	6.463	6.210	5.971	5.747	5.535	5.335	5.146	4.968	4.799	4.639	4.487	4.344	4.207	4.078	3.954	3.837
9	8.566	8.162	7.786	7.435	7.108	6.802	6.515	6.247	5.995	5.759	5.537	5.328	5.132	4.946	4.772	4.607	4.451	4.303	4.163	4.031
10	9.471	8.983	8.530	8.111	7.722	7.360	7.024	6.710	6.418	6.145	5.889	5.650	5.426	5.216	5.019	4.833	4.659	4.494	4.339	4.192
11	10.368	9.787	9.253	8.760	8.306	7.887	7.499	7.139	6.805	6.495	6.207	5.938	5.687	5.453	5.234	5.029	4.836	4.656	4.486	4.327
12	11.255	10.575	9.954	9.385	8.863	8.384	7.943	7.536	7.161	6.814	6.492	6.194	5.918	5.660	5.421	5.197	4.988	4.793	4.611	4.439
13	12.134	11.348	10.635	9.986	9.394	8.853	8.358	7.904	7.487	7.103	6.750	6.424	6.122	5.842	5.583	5.342	5.118	4.910	4.715	4.533
14	13.004	12.106	11.296	10.563	9.899	9.295	8.745	8.244	7.786	7.367	6.982	6.628	6.302	6.002	5.724	5.468	5.229	5.008	4.802	4.611
15	13.865	12.849	11.938	11.118	10.380	9.712	9.108	8.559	8.061	7.606	7.191	6.811	6.462	6.142	5.847	5.575	5.324	5.092	4.876	4.675
16	14.718	13.578	12.561	11.652	10.838	10.106	9.447	8.851	8.313	7.824	7.379	6.974	6.604	6.265	5.954	5.668	5.405	5.162	4.938	4.730
17	15.562	14.292	13.166	12.166	11.274	10.477	9.763	9.122	8.544	8.022	7.549	7.120	6.729	6.373	6.047	5.749	5.475	5.222	4.990	4.775
18	16.398	14.992	13.754	12.659	11.690	10.828	10.059	9.372	8.756	8.201	7.702	7.250	6.840	6.467	6.128	5.818	5.534	5.273	5.033	4.812
19	17.226	15.678	14.324	13.134	12.085	11.158	10.336	9.604	8.950	8.365	7.839	7.366	6.938	6.550	6.198	5.877	5.584	5.316	5.070	4.843
20	18.046	16.351	14.877	13.590	12.462	11.470	10.594	9.818	9.129	8.514	7.963	7.469	7.025	6.623	6.259	5.929	5.628	5.353	5.101	4.870
25	22.023	19.523	17.413	15.622	14.094	12.783	11.654	10.675	9.823	9.077	8.422	7.843	7.330	6.873	6.464	6.097	5.766	5.467	5.195	4.948
30	25.808	22.396	19.600	17.292	15.372	13.765	12.409	11.258	10.274	9.427	8.694	8.055	7.496	7.003	6.566	6.177	5.829	5.517	5.235	4.979
35	29.409	24.999	21.487	18.665	16.374	14.498	12.948	11.655	10.567	9.644	8.855	8.176	7.586	7.070	6.617	6.215	5.858	5.539	5.251	4.992
40	32.835	**27.355**	23.115	19.793	17.159	15.046	13.332	11.925	10.757	9.779	8.951	8.244	7.634	7.105	6.642	6.233	5.871	5.548	5.258	4.997
50	39.196	31.424	25.730	21.482	18.256	15.762	13.801	12.233	10.962	9.915	9.042	8.304	7.675	7.133	6.661	6.246	5.880	5.554	5.262	4.999

Example 3 cont

- T = 20
- $12{,}462 = 1000\left\{\frac{1-\left(\frac{1}{1+r}\right)^{20}}{r}\right\}$
- $\left\{\frac{1-\left(\frac{1}{1+r}\right)^{20}}{r}\right\} = 12.462$
- r = 5% per semi-annual (= 10% p.a.)

➔ Market discount rate/Yield = 10% p.a.

Example 4: coupon bond

- A 3-year government bond is set to pay the coupon for every six months, with coupon rate of 6% pa. Calculate the fair price if the market rate is 4% pa and the face value of bond is $100.

Example 4: coupon bond

- Periods before it is matured= ________ periods
- Cash flow = ________ per period
- Discount rate = __________

$$V_t \quad =$$

Example 4

- PVIFA(______%, _______) = ________

- $PVF(\_\_\_\_\_\_\%,\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_$

- Value =

Example 3 cont

- What if c = 2% and 4%?

$$Coupon = 2\,\% \Rightarrow 1\left\{\frac{\left(\frac{1}{1+0.02}\right)-\left(\frac{1}{1+0.02}\right)^{6+1}}{1-\left(\frac{1}{1+0.02}\right)}\right\}+\frac{100}{(1+0.02)^6}=94.48$$

$$Coupon = 4\,\%: \Rightarrow 2\left\{\frac{\left(\frac{1}{1+0.02}\right)-\left(\frac{1}{1+0.02}\right)^{6+1}}{1-\left(\frac{1}{1+0.02}\right)}\right\}+\frac{100}{(1+0.02)^6}=100$$

Observations from the Bond pricing formula

$$P > FV \quad sold\ at\ Premium \quad yield < coupon$$
$$P = FV \quad sold\ at\ PAR \quad yield = coupon$$
$$P < FV \quad sold\ at\ Discount \quad yield > coupon$$

Sum up…

- Bond pricing formula
- In the equilibrium, the formula can be applied to calculate
  - Fair price / Market price
  - Implied "yield to maturity/interest rate"
- Relationship between bond price & FV and Interest rate & Coupon

Topics

- Bond valuation and Yield to maturity
- Rate of return on bond investment

Yield V.S. Rate of return (ROR)

- Suppose the investment horizon is 1 period (t → t+1)
- The bond pays c% of FV.
- Definition: rate of return on one-period investment on bond is

$$RER_t = \frac{P_{t+1} + \frac{c}{100}FV - P_t}{P_t}$$

$$= \frac{P_{t+1} - P_t}{P_t} + \frac{\frac{c}{100}FV}{P_t}$$

Yield V.S. Rate of return (ROR)

- $\frac{\frac{c}{100}FV}{P_t}$ = current yield from coupon
- $\frac{P_{t+1}-P_t}{P_t}$ = ??? ➔ P_{t+1} =????
  - P_{t+1} is uncertain! :
  - "+ Cap Gain", "- Cap Loss"
  - Fluctuations in bond price could arise due to several reasons, associated with risk of bond holdings.

Bond risks

- Market-price risk
- Credit risk
- Inflation risk

Three main risk factors for Bond price volatilities

- Event risk
- Legal risk
- Exchange rate risk
- Etc….

Market-price risk

- Following the bond pricing, we know that

$$P_t = V_t = \frac{CF_1}{1 + r_T} + \frac{CF_2}{(1 + r_T)^2} + \cdots + \frac{CF_T}{(1 + r_T)^T}$$

- How does r_T affect the price?
  - First observation: price can be volatile due to the change in the market sentiment/trend of interest rate.
- Volatility in price due to the change in interest rate is called "interest rate risk".

Example: Interest rate risk

- Fund manager is considering to purchase a bond with the following details.

A 2-year bond with $100 Face value is set to pay 1% of coupon rate once a year. Suppose that current market interest rate is 1%

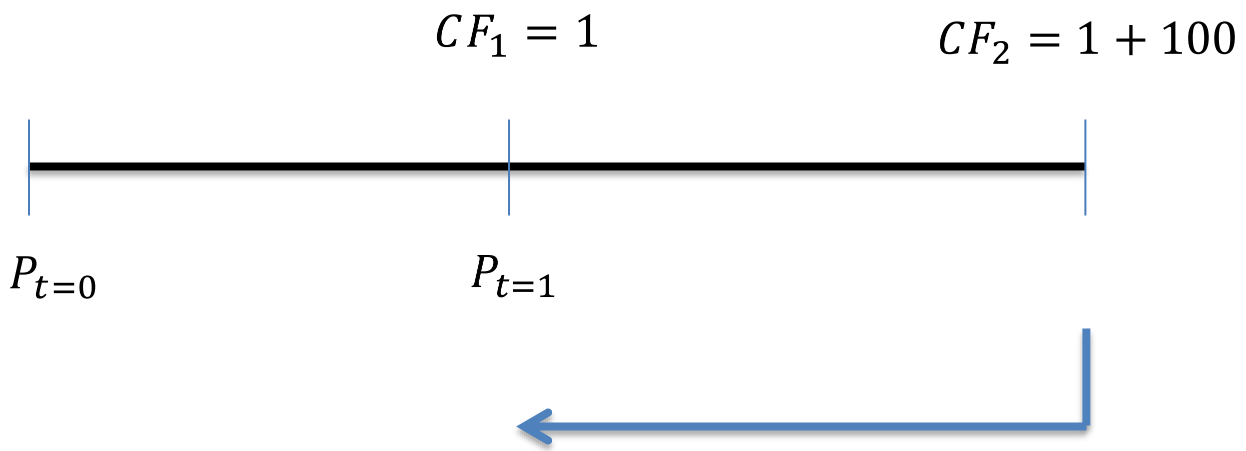
- What is the current market price?

Example: Interest rate risk

Suppose that market interest rate is expected to be 2% next year.

- Calculate bond price in the next year. What is the rate of return from the investment?

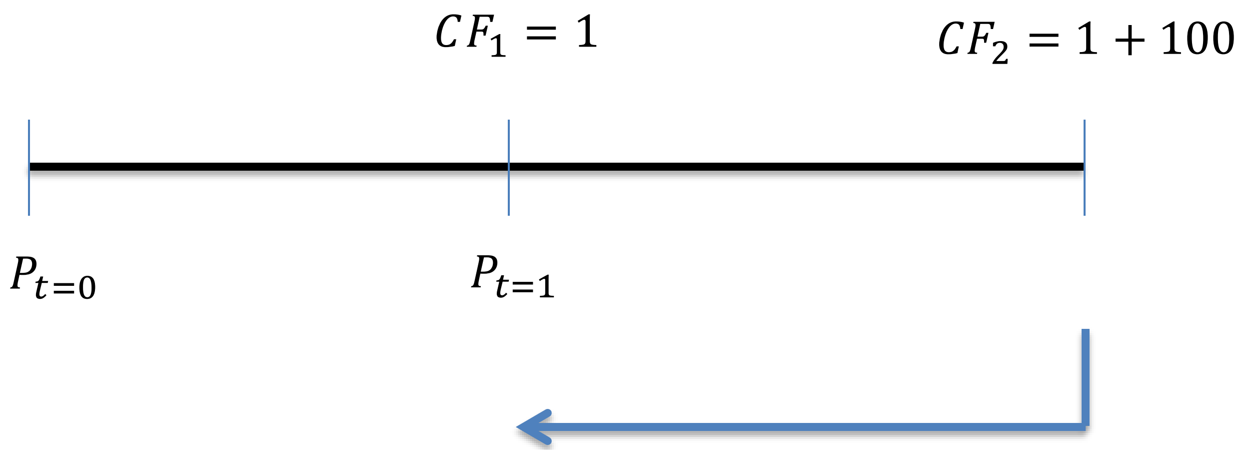
Example: Interest rate risk



Price at time t = 1

$$V_{t=1} = \frac{(1 + 100)}{(1 + 0.02)} = 99.01$$

Example: Interest rate risk



$$P_{t=1} = 99.01 \Rightarrow Cap\ Loss = \_\_\_\_\_\_\_\_\_\_.$$

Rate of return = Current yield + Cap Loss

Example

The same fund manager considers changing his investment plan to long-term investment.

- An 11-year bond with Coupon = 1%, market interest rate is 1%

What would be the rate of return if the interest rate increases to 2% next year.

Example: interest rate risk

- Current price ____________________
- Price in next year if interest rate increases to 2%

$$P_{t=1} = 1 * PVIFA(2\%, 10) + 100 * PVF(2\%, 10)$$

- Price risk =
- Rate of return =

Behavior of interest rate risk

Depends on…

1. Price (interest rate) risk “increases” with “Time to maturity: TTM”.

Behavior of interest rate risk

Table 2 One-Year Returns on Different-Maturity 10%-Coupon-Rate Bonds When Interest Rates Rise from 10% to 20%

(1) Years to Maturity When Bond Is Purchased	(2) Initial Current Yield (%)	(3) Initial Price ($)	(4) Price Next Year* ($)	(5) Rate of Capital Gain (%)	(6) Rate of Return (2 + 5) (%)
30	10	1,000	503	−49.7	−39.7
20	10	1,000	516	−48.4	−38.4
10	10	1,000	597	−40.3	−30.3
5	10	1,000	741	−25.9	−15.9
2	10	1,000	917	−8.3	+1.7
1	10	1,000	1,000	0.0	+10.0

*Calculated using Equation 3.

Measuring the interest rate risk

- Quant finance has developed some indicators measuring the interest rate.

- A commonly used risk measure for bond (interest rate risk) is the Macaulay Duration (MacD)
  - Frederick Macaulay (1938)

Measuring the interest rate risk: MacD

- Suppose that bond is matured T-period from now.

MacD = $\sum_{t=1}^{T} t \frac{PV_t}{Price} = 1\frac{PV_1}{P} + 2\frac{PV_2}{P} + \cdots + T\frac{PV_T}{P}$

  - PV_t = present value of cash flow receivable at time "t"
  - Price = current price

Measuring the interest rate risk

Higher MacD → higher Interest rate risk

- Following the MacD formula, the value varies with "T"

Measuring the interest rate risk

$$\text{MacD} = 1\frac{PV_1}{P} + 2\frac{PV_2}{P} + \cdots + T\frac{PV_T}{P}$$

One can see that $\frac{PV_T}{Price} = \frac{C+FV}{price}$ is the largest term!

Therefore, "T" will be receiving the highest weight when one sums up the number of periods of payment.

Measuring the interest rate risk

- What is MacD / Bond duration in economics?
- What does it mean in terms of quantitative implications?
  - MacD is an input of the linear-approximation formula that is used to measure/estimate the % change in bond price, with respect to market interest rate.

Measuring the interest rate risk

Following the Bond pricing:

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C+FV}{(1+r)^T}$$

$$\frac{dP}{dr} = -1\frac{C}{(1+r)^2} - 2\frac{C}{(1+r)^3} + \cdots - T\frac{C+FV}{(1+r)^{T+1}}$$

$$\frac{dP}{dr} = \left\{-1\frac{C}{(1+r)} - 2\frac{C}{(1+r)^2} + \cdots - T\frac{C+FV}{(1+r)^T}\right\} \times \frac{1}{1+r}$$

$$\frac{dP}{dr} = (-1 * PV_1 - 2 * PV_2 + \cdots - T * PV_T) \times \frac{1}{1+r}$$

Measuring the interest rate risk

$$dP = (-1 * PV_1 - 2 * PV_2 + \cdots - T * PV_T) \times \frac{dr}{1+r}$$

$$\frac{dP}{P} = \left(-1 * \frac{PV_1}{P} - 2 * \frac{PV_2}{P} + \cdots - T * \frac{PV_T}{P}\right) \times \frac{dr}{1+r}$$

$$\frac{dP}{P} = -\frac{(MacD)}{1+r} dr$$

*negative sing → inverse relationship between "r" and "P"

Measuring the interest rate risk

$$\frac{dP}{P} = -\frac{(MacD)}{1+r}dr = -(ModD)dr$$

$(ModD)$ = modified duration

Measuring the % change in price for a 100 basis points change in interest rate

Higher ModD → larger variations in price.

Inputs?

Example

Coupon rate	=	8.00%
Term	=	5 years (semi-annual interest payment)
Yield-to-maturity	=	8.00%
Price	=	100

Period (t)	Cash flow	PVCF	t x $PVCF_t$
1	$4.0	3.8462	3.8462
2	4.0	3.6982	7.3964
3	4.0	3.5560	10.6680
4	4.0	3.4192	13.6769
5	4.0	3.2877	16.4385
6	4.0	3.1613	18.9675
7	4.0	3.0397	21.2777
8	4.0	2.9228	23.3821
9	4.0	2.8103	25.2931
10	104.0	70.2586	702.5867
		-------------	------------
Total		100.0000	843.5331

Macaulay duration =

Example

- Following the table: Calculate the % change in price if the market interest rate drops from 8% to 7%

$$\frac{(MacD)}{1+r}dr$$

$1 + r =$ ______________

$dr =$ ________________

Behavior of interest rate

$$\frac{(MacD \uparrow)}{1+r} \Rightarrow \boldsymbol{ModD} \uparrow$$

- We already knew that long-term bond contains high price risk.

- $MacD$ increases with TTM ➔ higher Interest rate risk

Behavior of interest rate

$$\frac{(MacD)}{1+r\downarrow} \Rightarrow ModD\ \uparrow$$

- We already knew that long-term bond contains high price risk.
- $MacD$ increases with TTM ➔ higher Interest rate risk
- For a fixed "T", lower "r" implies higher interest rate risk!

Behavior of interest rate risk

Depends on…

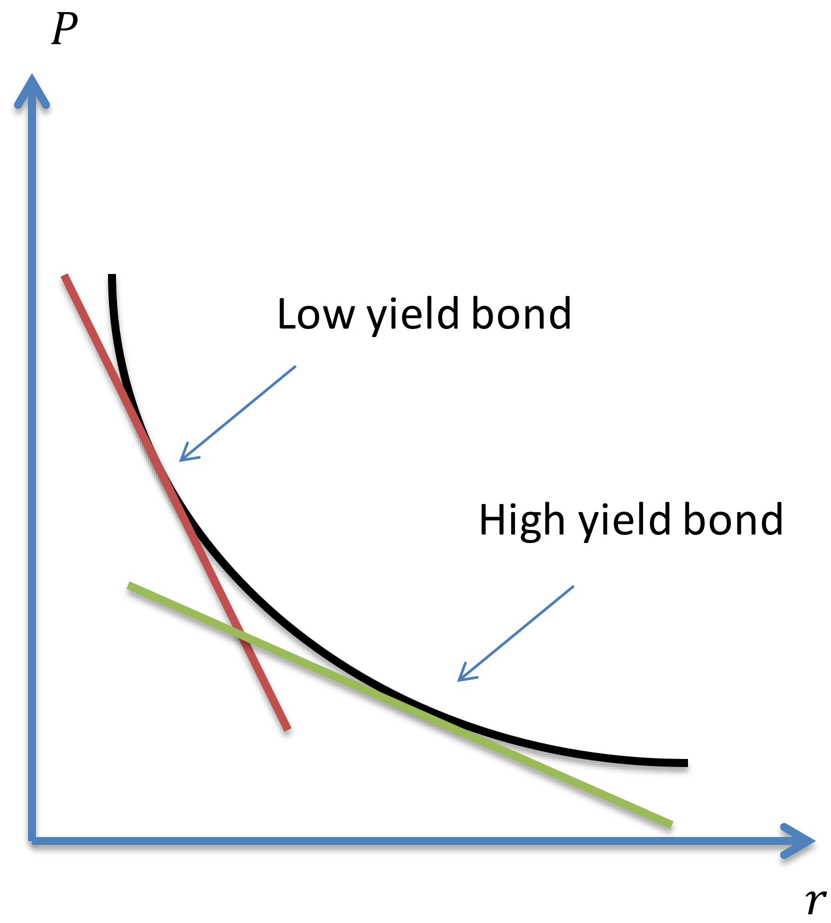
1. Price (interest rate) risk “increases” with “Time to maturity: TTM”.
2. Price (interest rate) risk negatively varies with the initial level of interest rate.

Why Convexity?

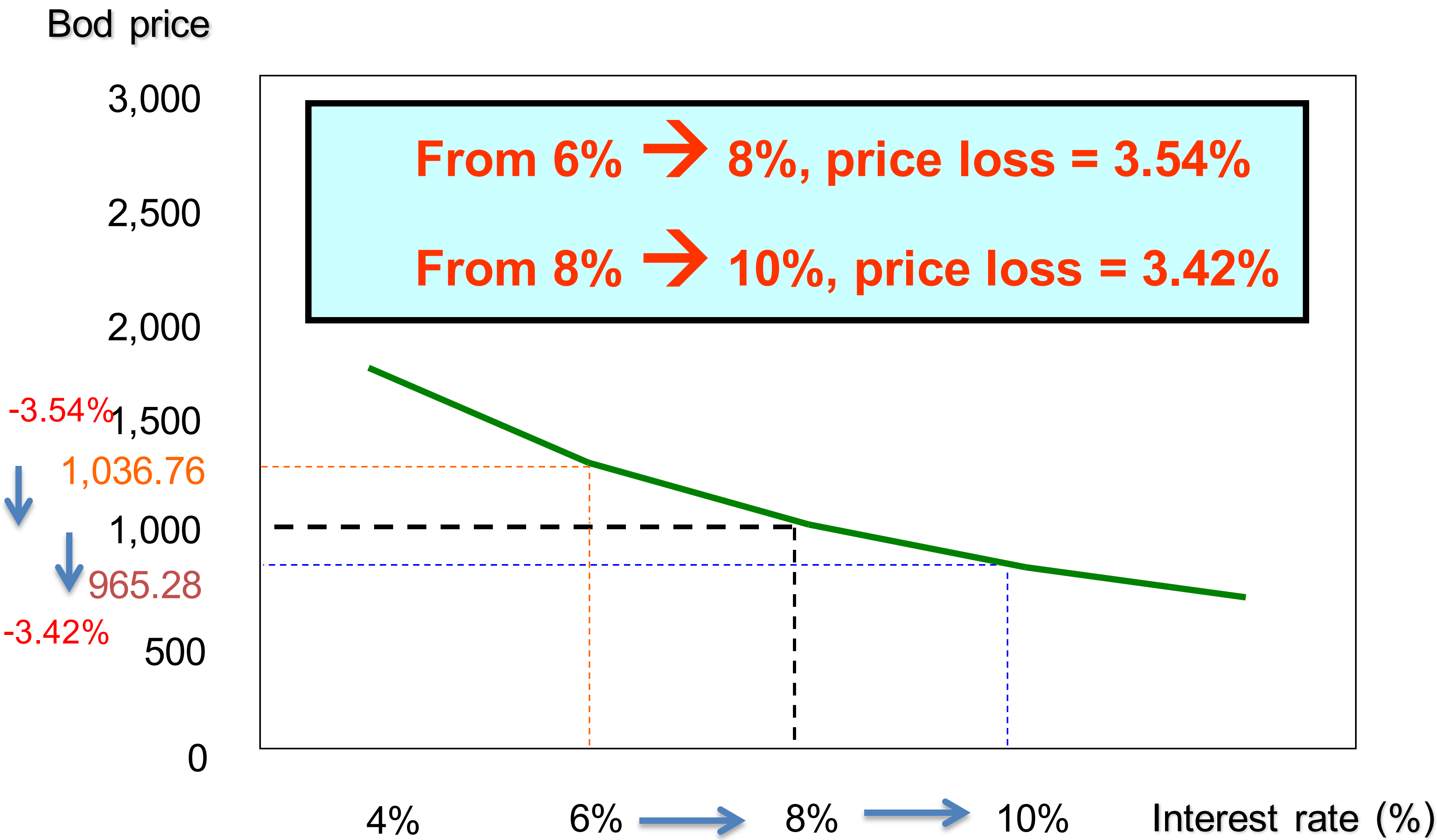
- Again, bond pricing

$$P = \frac{1}{(1+r)^T}$$

- Hyperbolic function
  - Convex function
  - $\frac{d^2P}{dr^2} < 0$



Suppose Coupon rate = 8% and FV = 1000.



Limitations of the duration concept as the measurement of interest rate risk

1. The formula stems from a ***linear*** approximation!

   - Linear is the size of change of interest rate.

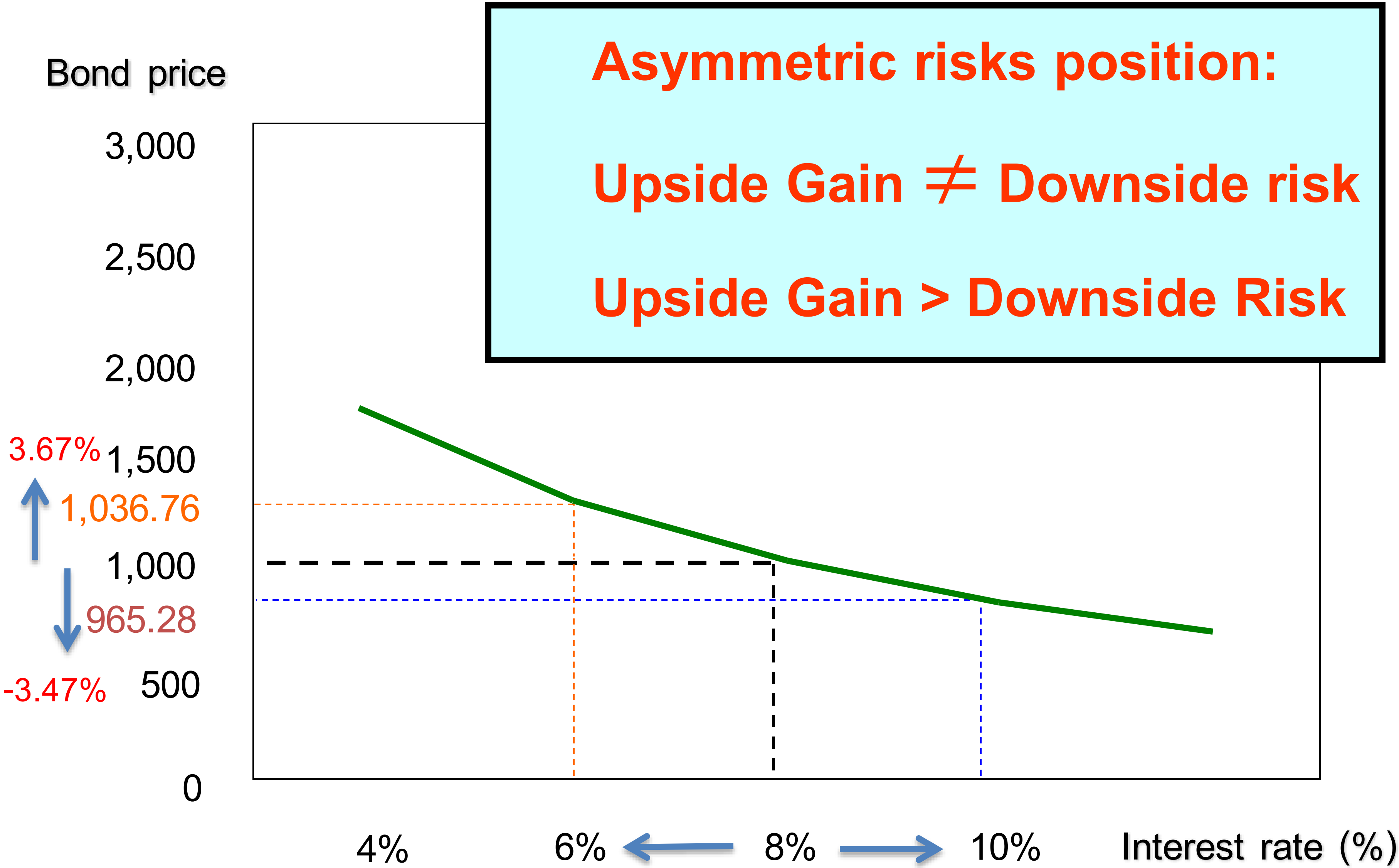
   - Good approximate for daily trade when yield does not change dramatically

     (small change in yield.)

2. Duration predicts equal upside gain/downsize loss.

   - Bond risk has the *asymmetric* feature.

Coupon rate = 8% and FV = 1000.

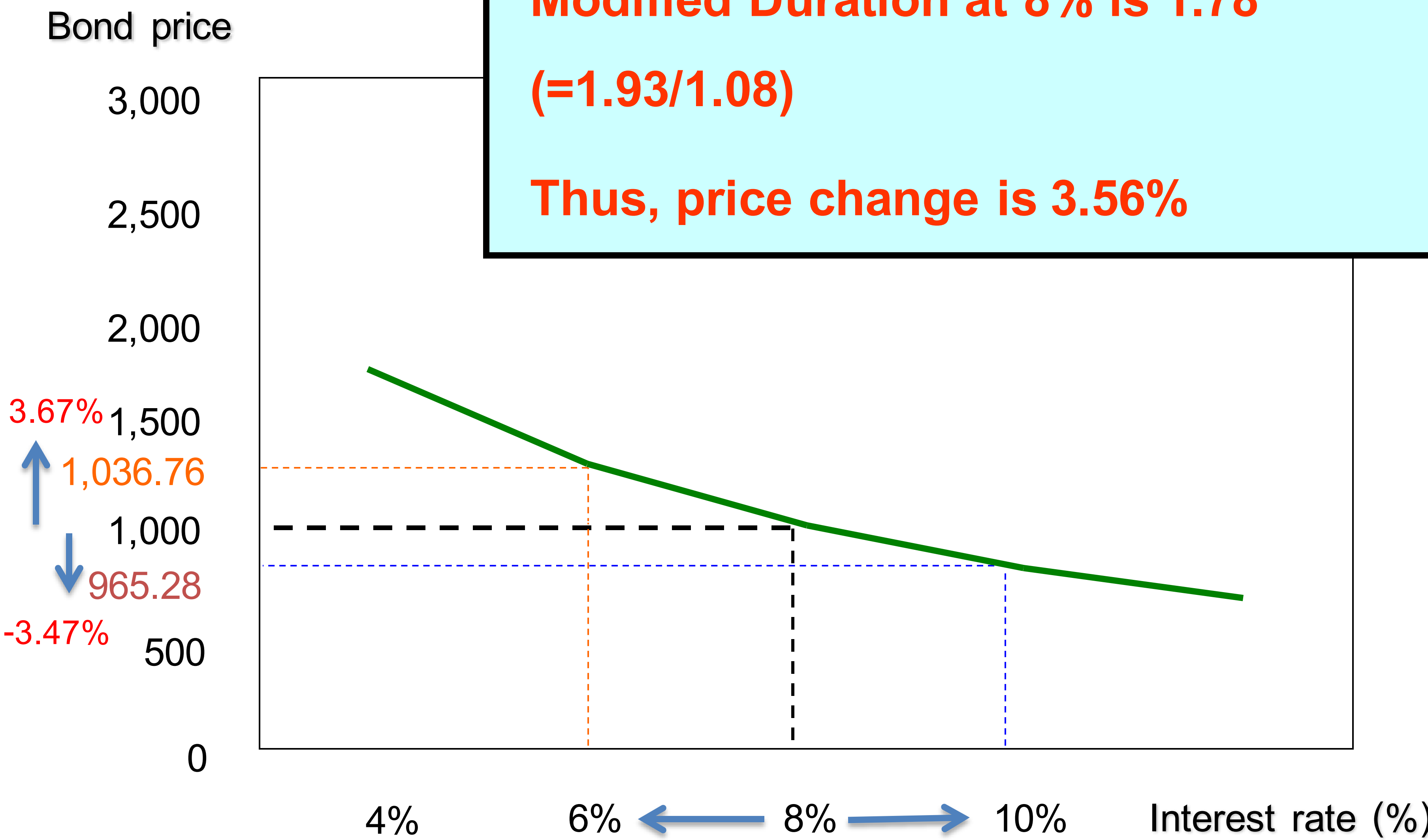


Accuracy of *Duration* in measuring risks

Modified Duration at 8% is 1.78

(=1.93/1.08)

Thus, price change is 3.56%

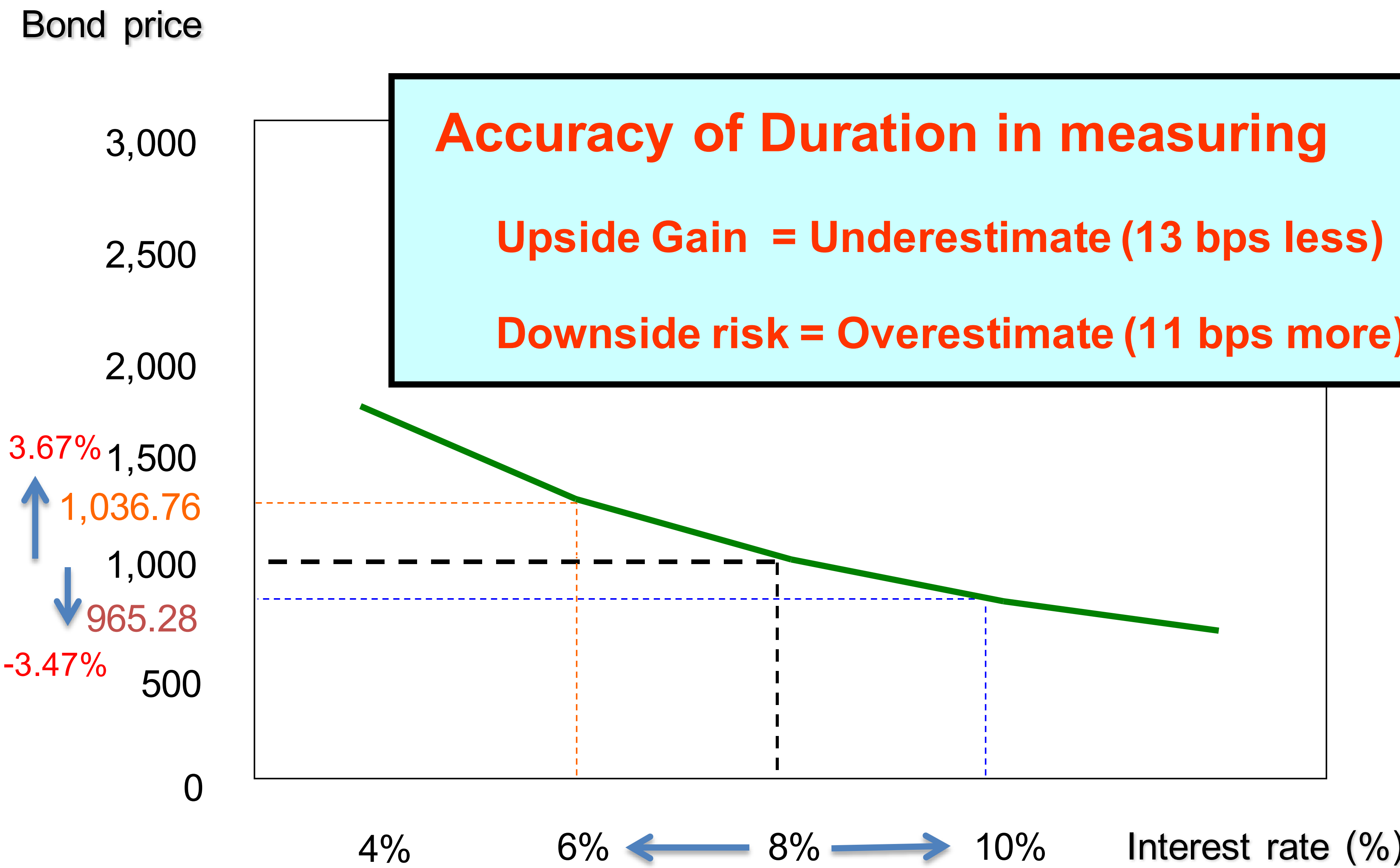


Accuracy of *Duration* in measuring risks

Accuracy of Duration in measuring

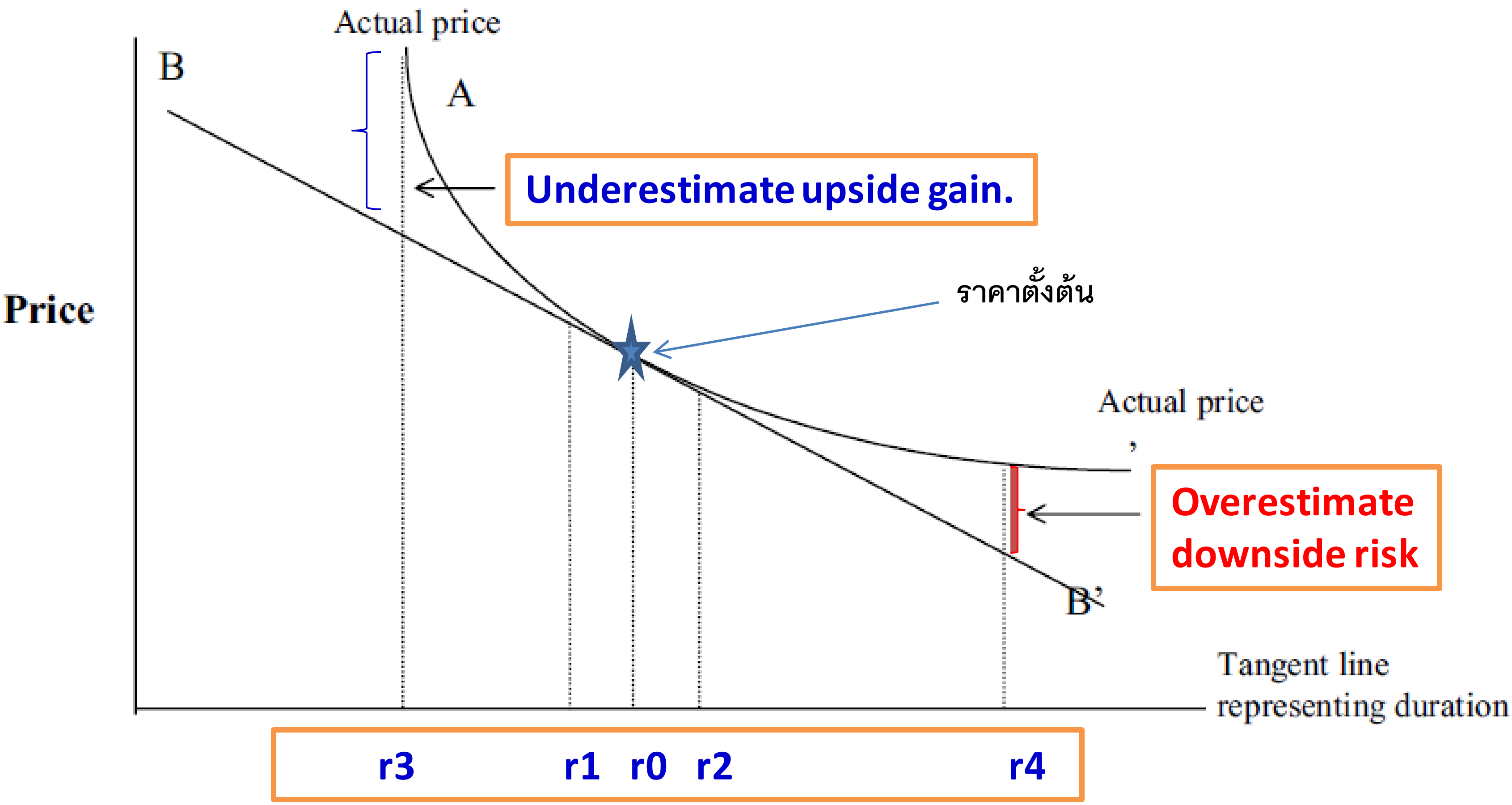
Upside Gain = Underestimate (13 bps less)

Downside risk = Overestimate (11 bps more)



Convexity and Pricing errors

- Error in risk calculation: Slope of BB’ is the Modified duration



Digression: approximation theory

General theory:

$$f(x)$$
$$= \mathrm{f}(\mathrm{x}_0) + \mathrm{f}'(\mathrm{x}_0)(\mathrm{x} - \mathrm{x}_0) + \frac{1}{2}\mathrm{f}''(\mathrm{x}_0)(\mathrm{x} - \mathrm{x}_0)^2$$

Bond pricing approximation

Now tread the bond price formula as the function of "r"

$$P = f(r) = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C+FV}{(1+r)^T}$$

The apply the Second-order approximation

$$dP = \frac{dP}{dr} * (r - r_0) + \frac{1}{2}\frac{d^2p}{dr^2} * (r - r_0)^2$$

Bond pricing approximation

$$\frac{dP}{P} = \frac{dP}{dr} * \frac{1}{P} * (r - r_0) + \frac{1}{2}\frac{d^2p}{dr^2} * \frac{1}{P} * (r - r_0)^2$$

$$\frac{dP}{dr} * \frac{1}{P} = \frac{(MacD)}{1 + r_0}$$

$$\frac{d^2P}{dr^2} * \frac{1}{P} = \text{convexity}$$

Bond pricing approximation: convexity formula

- Convexity = $\frac{\sum_{t=1}^{T} t(t+1)\frac{PV_t}{Price}}{(1+r_0)^2}$

$$= \frac{\left[1(2)\frac{PV_1}{P} + 2(3)\frac{PV_2}{P} + \cdots + T(T+1)\frac{PV_T}{P}\right]}{(1+r_0)^2}$$

Example: Upside gain

Coupon rate	=	8.00%	
Term	=	5 years	(semi-annual interest payment)
Yield-to-maturity	=	8.00%	
Price	=	100	

Period (t)	Cash flow	PVCF	t x $PVCF_t$
1	$4.0	3.8462	3.8462
2	4.0	3.6982	7.3964
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8	4.0	2.9228	23.3821
9	4.0	2.8103	25.2931
10	104.0	70.2586	702.5867
		-------------	------------
Total		100.0000	843.5331

Macaulay duration =

Linear approximation

4.055%

Non-linear approximation

4.055% + 0.1009%

Upside gain = 4.1564

Example: Downside risk

Coupon rate	=	8.00%
Term	=	5 years (semi-annual interest payment)
Yield-to-maturity	=	8.00%
Price	=	100

Period (t)	Cash flow	PVCF	t x $PVCF_t$
1	$4.0	3.8462	3.8462
2	4.0	3.6982	7.3964
3	4.0	3.5560	10.6680
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8	4.0	2.9228	23.3821
9	4.0	2.8103	25.2931
10	104.0	70.2586	702.5867
		------------	------------
Total		100.0000	843.5331

Macaulay duration =

Linear approximation

4.055%

Non-linear approximation

-4.055% + 0.1009%

Upside gain = -3.9546%

Conclusion

- Understanding basic bond pricing formula
- Measuring market-price risk of bond
- Duration as the approximation of interest rate risk
- Convexity gives a more accurate approximate to the risk, both upside and downside perspective.