Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score......................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall t} \ E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income ($y_t = 0\ \forall t$) and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

$$\max_{C_s \omega_s \forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Please read and answer the following questions carefully and completely.

Score......................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

Let $S_t = W_t - C$ $\quad R_t : R_t \equiv R_f + \omega^*_t (R_{rt} - R_f)$

$$U_C = E_{T-1}[B_U R_{T-1}]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}[\delta^T W_T^{-\gamma} R_{T-1}]$$

$$= E_{T-1}[\delta^T (S_{T-1} R_{T-1})^{-\gamma} R_{T-1}]$$

$$= \delta^T E_{T-1}[R^{1-\gamma}](W_{T-1} - C_{T-1})^{-\gamma}$$

or

$$C_{T-1} = \frac{(\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{\frac{1}{-\gamma}}}{1 + (\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{\frac{1}{-\gamma}}} W_{T-1}$$

$$= \frac{a_1}{1+a_1} W_{T-1} = C_1 W_{T-1}$$

where $C_1 = \frac{a_1}{(1+a_1)}$ $\quad a_1 \equiv (\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{\frac{1}{-\gamma}} = (\delta E[R_{T-1}^{1-\gamma}])^{\frac{1}{-\gamma}}$

Please read and answer the following questions carefully and completely.

Score......................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$E_{T-1}[B_\omega R_{r,T-1}] = R_f E_{T-1}[B_\omega]$$

$$\delta^T E_{T-1}[(S_{T-1} R_{T-1})^{-\gamma} R_{r,T-1}] = \delta^T R_{f,T-1} E_{T-1}[(S_{T-1} R_{T-1})^{\gamma-1}]$$

$$E_{T-1}[R_{T-1}^{-\gamma} R_{r,T-1}] = R_{f,T-1} E_{T-1}[R_{T-1}^{-\gamma}]$$

$$E[R_{T-1}^{\gamma-1} R_{r,T-1}] = R_{f,T-1} E[R_{T-1}^{-\gamma}]$$

Score......................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1},T-1) = \delta^{T-1} C_{T-1}^{*1-\gamma}/1-\gamma + \delta^{T} E_{T-1}\left[\frac{\left(R^*_{T-1}(W_{T-1}-C^*_{T-1})\right)^{1-\gamma}}{1-\gamma}\right]$$

$$= \delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^{1-\gamma}\frac{W_{T-1}^{1-\gamma}}{1-\gamma} + \delta^{T} E_{T-1}\left[R_{T-1}^{1-\gamma}\frac{W_{T-1}^{1-\gamma}}{(1-\gamma)(1+a_1)^{1-\gamma}}\right]$$

$$= \delta^{T-1}\frac{W_{T-1}^{1-\gamma}}{(1-\gamma)(1+a_1)^{1-\gamma}}\left(a_1^{1-\gamma} + \delta E_{T-1}\left[R_{T-1}^{1-\gamma}\right]\right)$$

$$= \delta^{T-1}\frac{b_1 W_{T-1}^{1-\gamma}}{1-\gamma}$$

where $b_1 = \frac{\left(a_1^{1-\gamma} + \delta E\left[R_{T-1}^{1-\gamma}\right]\right)}{(1+a_1)^{1-\gamma}} = \left[\frac{a_1}{(1+a_1)}\right]^{-\gamma}$

Score…………………

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$U_C(C^*_{T-2}, T-2) = E_{T-2}\left[J_W(W_{T-1}, T-1) R_{T-2}\right]$$

$$C_{T-2}^{-\gamma} = \delta^{T-1} E_{T-2}\left[b_1 W_{T-1}^{-\gamma} R_{T-2}\right]$$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}\left[b_1 (S_{T-2} R_{T-2})^{-\gamma} R_{T-2}\right]$$

$$= \delta b_1 E_{T-2}\left[R_{T-2}^{1-\gamma}\right] (W_{T-2} - C_{T-2})^{-\gamma}$$

or $$C^*_{T-2} = \frac{\left(\delta b_1 E_{T-2}\left[R_{T-2}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}}{1 + \left(\delta b_1 E_{T-1}\left[R_{T-2}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}} W_{T-2}$$

$$= \frac{\partial_2}{1+\partial_2} W_{T-2}$$

$$= C_2 W_{T-2}$$

where $C_2 = \frac{\partial_2}{(1+\partial_2)}$ $\quad \partial_2 = \left(b_1 \delta E\left[R_{T-2}^{1-\gamma}\right]\right)^{\frac{1}{-\gamma}}$

$$= \partial_1^2 / (1+\partial_1)$$

$$= \partial_1 C_1$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$E_{T-2}\left[J_W R_{r,T-2}\right] = R_f E_{T-2}\left[J_W\right]$$

$$E_{T-2}\left[b_1 (S_{T-2} R_{T-2})^{-\gamma} R_{r,T-2}\right] = R_f E_{T-1}\left[b_1 (S_{T-2} R_{T-2})^{-\gamma}\right]$$

$$E_{T-2}\left[b_1 R_{T-2}^{-\gamma} R_{r,T-2}\right] = R_f E_{T-2}\left[b_1 R_{T-2}^{-\gamma}\right]$$

$$E\left[R_{T-2}^{-\gamma} R_{r,T-2}\right] = R_f E\left[R_{T-2}^{-\gamma}\right]$$

Score......................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2},T-2) = U(C^*_{T-2},T-2) + E_{T-1}[J(W_{T-1},T-1)]$$

$$= \delta^{T-2}\frac{C^{*\,1-\gamma}_{T-2}}{1-\gamma} + E_{T-1}\left[\delta^{T-1}E_{T-1}\left[b_1\frac{(R^*_{T-2}(W_{T-2}-C^*_{T-2}))^{1-\gamma}}{1-\gamma}\right]\right]$$

$$= \delta^{T-2}\left(\frac{a_2}{1+a_2}\right)^{1-\gamma}\frac{W^{1-\gamma}_{T-2}}{1-\gamma} + \delta^{T-1}E_{T-1}\left[b_1\frac{(R_{T-2}(W_{T-2}-C^*_{T-2}))^{1-\gamma}}{1-\gamma}\right]$$

$$= \delta^{T-2}\left(\frac{a_2}{1+a_2}\right)^{1-\gamma}\frac{W^{1-\gamma}_{T-2}}{1-\gamma} + \delta^{T-1}E_{T-1}\left[b_1 R^{*\,1-\gamma}_{T-2}\frac{W^{1-\gamma}_{T-2}}{1-\gamma(1+a_2)^{1-\gamma}}\right]$$

$$= \delta^{T-2}\frac{W^{1-\gamma}_{T-2}}{1-\gamma(1+a_2)^{1-\gamma}}\left(a_2^{1-\gamma} + \delta E_{T-2}[R^{1-\gamma}_{T-2}]\right)$$

$$= \delta^{T-1} b_2 \frac{W^{1-\gamma}_{T-2}}{1-\gamma}$$

where $b_2 = \dfrac{\left(a_2^{1-\gamma} + \delta E[R^{*\,1-\gamma}_{T-2}]\right)}{(1+a_2)^{1-\gamma}}$

$E[R^{-\gamma}_{T-2} R_{f,T-2}] = R_f E[R^{-\gamma}_{T-2}]$

the level of consumption at T-1 is $C^*_{T-2} = C_2 W_{T-2}$ where $C_2 = a_1 C_1/(1+a_1 C_1)$

hence $C^*_{T-3} = C_3 W_{T-3}$ where $C_3 = \dfrac{a_1 C_2}{1+a_1 C_2}$ $C_{T-t} = C_t W_{T-t}$ where $C_t = \dfrac{a_1 C_{t-1}}{(1+a_1 C_{t-1}}$