

# Optimization with Equality Constraints I

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CW Ch.12

# Outline

- Effects of a Constraint
- Finding the Stationary Values
- Second-Order Conditions\*
- Utility Maximization and Consumer Demand\*
- Homogeneous Functions\*
- Least-Cost Combination of Inputs\*

# Effect of a Constraint

- Consider a consumer with the simple utility function

$$U = x_1x_2 + 2x_1$$

- Observe that the marginal utilities (partial derivative  $U_1 = \frac{\partial U}{\partial x_1}$  and

$$U_2 = \frac{\partial U}{\partial x_2})$$
 are positive for all positive levels of  $x_1$  and  $x_2$ .

- To have  $U$  maximized without any constraint, the consumer should purchase an *infinite* amount of both goods, a solution that has little practical relevance.

# Effect of a Constraint

- To make the optimization problem meaningful, the ***budget constraint*** is taken into account.
- Suppose the current prices of the two goods are  $P_{10} = 4$  and  $P_{20} = 2$  and the total budget of the consumer is 60, then the budget constraint can be expressed as

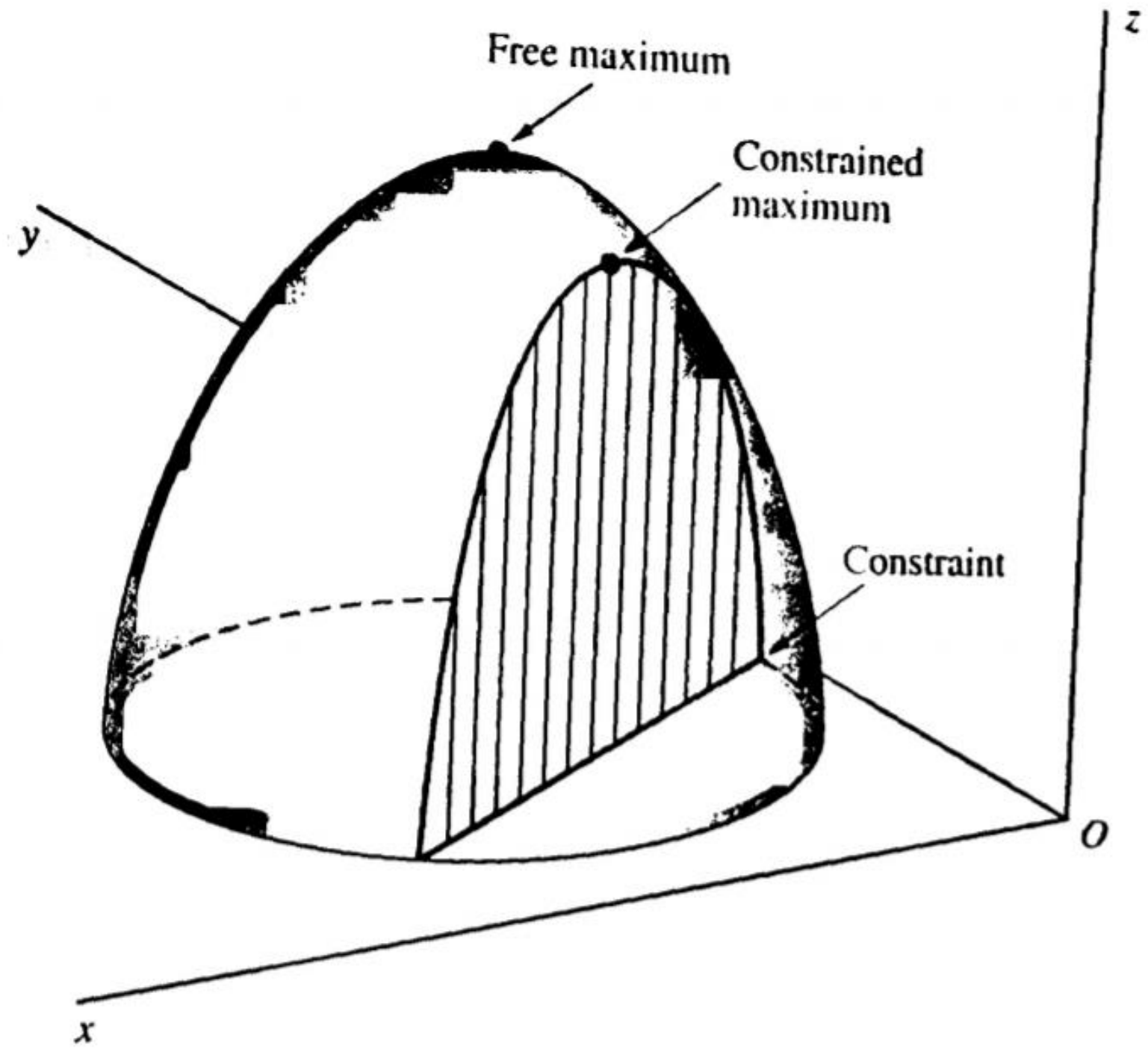
$$4x_1 + 2x_2 = 60.$$

- The constraint creates a tradeoff. If the consumer wants to increase his consumption of  $x_1$ , he has to reduce his consumption of  $x_2$ .
- Mathematically, what the constraint does is to narrow the domain, and hence the range of the objective function.

# Effect of a Constraint

- In our example, after the budget constraint is added, we can admit only  $(x_1, x_2)$  that satisfies the equation  $4x_1 + 2x_2 = 60$ .
- So that the domain is immediately reduced to the set of points lying on the budget line.
- This will automatically affect the range of the objective function, too. Only the subset of the utility values lying directly above the budget constraint will now be relevant.
- We will pick the maximum utility value from this restricted set.

# Effect of a Constraint



# Effect of a Constraint

- In general, for a function  $z = f(x, y)$ , the free extremum is the peak point, but the constrained extremum is at the peak of the curve situated on top of the constraint line.
- A constrained maximum can be expected to have a *lower* value than the free maximum, although, by coincidence, the two maxima may happen to have the same value.
- The constrained maximum can never exceed the free maximum.

# Finding the Stationary Values

- Even without any new technique, the constrained maximum in the previous example can be found.
- From the budget constraint, we have

$$x_2 = \frac{60 - 4x_1}{2} = 30 - 2x_1.$$

- Substitute this into the objective function, we get

$$U = x_1(30 - 2x_1) + 2x_1 = 32x_1 - 2x_1^2.$$

- FOC:

$$\frac{dU}{dx_1} = 32 - 4x_1 = 0,$$

which implies  $x_1^* = 8$ . Then we have  $x_2^* = 30 - 2(8) = 14$ , and  $U^* = 128$ .

# Finding the Stationary Values: Lagrange-Multiplier Method

- When the constraint is a complicated function, the previous technique of elimination of variables could be burdensome.
- We may switch to the method of **Lagrange Multiplier**.
- Given the problem of

$$\begin{aligned} \max_{x_1, x_2} U &= x_1 x_2 + 2x_1 \\ \text{s. t.} \quad &4x_1 + 2x_2 = 60, \end{aligned}$$

let the **Lagrangian equation** be

$$L = x_1 x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2),$$

where  $\lambda$  is called a **Lagrange multiplier**.

# Finding the Stationary Values: Lagrange-Multiplier Method

- We treat  $\lambda$  as an additional choice variable.
- FOC:

$$L_1 \left( \equiv \frac{\partial L}{\partial x_1} \right) = x_2 + 2 - 4\lambda = 0$$

$$L_2 \left( \equiv \frac{\partial L}{\partial x_2} \right) = x_1 - 2\lambda = 0$$

$$L_\lambda \left( \equiv \frac{\partial L}{\partial \lambda} \right) = 60 - 4x_1 - 2x_2 = 0.$$

The last equation guarantee the satisfaction of the constraint.

- Solving the above system of equations, we find  $x_1^* = 8$ ,  $x_2^* = 14$ , and  $\lambda^* = 4$ .
- The value of  $L^* = 128$ , which is the same as  $U^*$  from the previous method. Why?

# Finding the Stationary Values: Lagrange-Multiplier Method

- In general, consider the problem

$$\begin{aligned} \max_{x,y} z &= f(x, y) \\ \text{s. t. } g(x, y) &= c, \end{aligned}$$

where  $c$  is a constant.

- We can write the Lagrangian equation as

$$L = f(x, y) + \lambda[c - g(x, y)].$$

- To solve for stationary value of  $L$ , the first-order (necessary) condition is

$$\begin{aligned} L_x &= f_x - \lambda g_x = 0 \\ L_y &= f_y - \lambda g_y = 0 \\ L_\lambda &= c - g(x, y) = 0. \end{aligned}$$

# Finding the Stationary Values: Lagrange-Multiplier Method

- Let  $x^*$ ,  $y^*$  and  $\lambda^*$  be the solution of the problem. This means

$$f_x(x^*, y^*) - \lambda^* g_x(x^*, y^*) = 0$$

$$f_y(x^*, y^*) - \lambda^* g_y(x^*, y^*) = 0$$

$$c - g(x^*, y^*) = 0.$$

Rearrange the first two equations, we get

$$\frac{f_x(x^*, y^*)}{g_x(x^*, y^*)} = \lambda^* = \frac{f_y(x^*, y^*)}{g_y(x^*, y^*)}.$$

This is the optimal condition for constrained optimization.

# Finding the Stationary Values: Lagrange-Multiplier Method

- Now, consider the optimal value  $L^*$ , which comes from

$$L^* = f(x^*, y^*) + \lambda^*[c - g(x^*, y^*)].$$

Differentiating  $L^*$  with respect to  $c$ , we get

$$\begin{aligned}\frac{dL^*}{dc} &= f_x \frac{dx^*}{dc} + f_y \frac{dy^*}{dc} + [c - g(x^*, y^*)] \frac{d\lambda^*}{dc} + \lambda^* \left( 1 - g_x \frac{dx^*}{dc} - g_y \frac{dy^*}{dc} \right) \\ &= (f_x - \lambda^* g_x) \frac{dx^*}{dc} + (f_y - \lambda^* g_y) \frac{dy^*}{dc} + [c - g(x^*, y^*)] \frac{d\lambda^*}{dc} + \lambda^*.\end{aligned}$$

But from FOC, the first three terms turn into zero. Therefore, we are left with

$$\frac{dL^*}{dc} = \lambda^*.$$

- Lagrange multiplier  $\lambda^*$  measures the sensitivity of  $L^*$  to changes in parameter  $c$ . This condition will be discussed in economics later.

# Example 1

- Solve the problem

$$\begin{aligned} \max_{x,y} z &= xy \\ \text{s. t. } x + y &= 6. \end{aligned}$$

# Example 2

- Solve the problem

$$\begin{aligned} \max_{x,y} z &= x_1^2 + x_2^2 \\ \text{s. t. } x_1 + 4x_2 &= 2. \end{aligned}$$

# Assignment 5

- Exercise 12.2 No.1 and 3.