Group 4
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Question 0:

Consider the function f defined for all (x,y) such that

$$f\,(x,y;a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1-a^3, a^2)$ is a stationary point of $f\,(x,y;a)$.

$$\frac{\partial f}{\partial x} = x - 1 + ay = 0$$
$$x = 1 - ay$$

$$\frac{\partial f}{\partial y} = ax - a - y^2 + 2a^2y = 0$$

$$a(1-ay) - a - y^2 + 2a^2y = 0$$
$$a - a^2y - a - y^2 + 2a^2y = 0$$
$$-y^2 + a^2y = 0$$
$$-y^2 = -a^2y$$
$$y^* = a^2 \#$$

$$x = 1 - ay$$
$$x = 1 - a(a^2)$$
$$x = 1 - a^3$$
$$x^* = 1 - a^3 \#$$

b. State the condition under which the above stationary point is a global maximum.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & 2a^2 - 2y \end{bmatrix}$$

$$|H_1| = 1$$
$$|H_2| = 1(2a^2 - 2y) - a^2$$
$$= 2a^2 - 2y - a^2$$
$$= a^2 - 2y$$

Global maximum means that always $|H_1| \geq 0$ and $|H_2| > 0$ for all value of x and y.
In this case $|H_1| > 0$ and $|H_2| < 0$ for some values of y, so $f(x,y;a)$ is inconclusive

c. Given that $G(a) = f\,(x^*, y^*; a)$, show the derivative of G with respect to a.

$$G(a) = \frac{1}{2}(1-a^3)^2 - (1-a^3) + a(a^2)((1-a^3)-1) - \frac{1}{3}(a^2)^3 + a^2(a^2)^2$$
$$\frac{1}{2}(1-a^3)^2 - (1-a^3) - a^6 - \frac{1}{3}a^6 + a^6$$
$$\frac{1}{2}(1-a^3)^2 - (1-a^3) - \frac{1}{3}a^6$$
$$\frac{1}{2}(1-a^3)^2 - 1 + a^3 - \frac{1}{3}a^6$$

$$\frac{\partial G(a)}{\partial a} = (1-a^3)(-3a^2) + 3a^2 - 2a^5$$
$$= -3a^2 + 3a^5 + 3a^2 - 2a^5$$
$$= a^5 \#$$

d. Calculate $\frac{\partial f\,(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1 - a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?

$$\frac{\partial f}{\partial a} = xy - y + 2ay^2$$

$$= (1-a^3)a^2 - a^2 + 2a(a^2)^2$$

$$= a^2 - a^5 - a^2 + 2a^5$$

$$= a^5$$

e. Determine the domain of (x,y) in the xy-plan where f is convex.

For the function f to be convex, Hessian matrix must be positive definite

From (b), $|H_1| = 1 > 0$

$|H_2| = -2y + a^2 > 0$

$-2y > -a^2$

$2y < a^2$

$y < \frac{a^2}{2}$

Domain $y = (-\infty, \frac{a^2}{2})$ #

$x = 1 - ay$

For $y = \frac{a^2}{2}$: $x = 1 - a(\frac{a^2}{2})$

$x = 1 - \frac{a^3}{2}$

Domain $x = (1 - \frac{a^3}{2}, \infty)$ #

Question 2: Suppose that there are three groups of people who take Sky train to commute in Bangkok. The first group is students (s), the second group is senior citizens (old), and the third group is working-aged people. The demand for each group is given by the following equations:

Demand of students: $P_s = 8 - \left(\frac{1}{2}\right) Q_s$

Demand of senior citizens: $P_{old} = 16 - 2Q_{old}$

Demand of working-aged people: $P_w = 20 - Q_w$

The Sky train operator has a constant marginal cost at MC = \$4, and total cost at TC = 4Q + 10. Consider the following problems.

a) Determine the profit-maximizing level of output/price under third-degree price discrimination. Calculate the level of maximized profit.

b) Confirm your result in (a) with the second-order derivative test.

c) Calculate (i) consumer surplus for each of the three groups of consumers, and (ii) producer surplus.

d) Calculate the optimal level of total output if the Sky train operator can practice the first-degree price discrimination for each group of the consumers in the market.

a) Student $\Pi_s(P,Q) = P_s Q_s - TC$

$= \left[8 - \frac{1}{2}(Q_s)\right] Q_s - (4Q_s + 10)$

$= 8Q_s - \frac{1}{2}Q_s^2 - 4Q_s - 10$

$= -\frac{1}{2}Q_s^2 + 4Q_s - 10$

F.O.C. $\frac{d\Pi}{dQ_s} = -Q_s + 4 = 0$

$Q_s = 4$

when $Q_s = 4$, So $P_s = 6$

$\Pi_s = -8 + 16 - 10 = -2$

Senior Citizens. $\Pi_{old}(P,Q) = P_{old} Q_{old} - TC$

$= [(16 - 2Q_{old})Q_{old}] - (4Q_{old} + 10)$

$= 16Q_{old} - 2Q_{old}^2 - 4Q_{old} - 10$

$= -2Q_{old}^2 + 12Q_{old} - 10$

F.O.C = $\frac{d\Pi}{dQ_{old}} = -4Q_{old} + 12 = 0$

$Q_{old} = 3$

when $Q_{old} = 3$, So $P_{old} = 10$

$\Pi_{old} = -18 + 36 - 10 = 8$

Working aged-people $\Pi_w(P,Q) = P_w Q_w - TC$

$= [(20 - Q_w) \cdot Q_w] - (4Q_w + 10)$

$= 20Q_w - Q_w^2 - 4Q_w - 10$

$= -Q_w^2 + 16Q_w - 10$

F.O.C $\frac{d\Pi}{dQ_w} = -2Q_w + 16 = 0$

$Q_w = 8$

when $Q_w = 8$, So $P_w = 12$

$\Pi_w = -64 + 128 - 10 = 54$

b) S.O.C. $\longrightarrow H \begin{bmatrix} \Pi_{ss} & \Pi_{so} & \Pi_{sw} \\ \Pi_{os} & \Pi_{oo} & \Pi_{ow} \\ \Pi_{wo} & \Pi_{ws} & \Pi_{ww} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$|H_1| < 0$, $|H_2| > 0$, $|H_3| < 0$

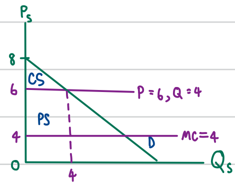
$d^2\Pi < 0$ for $V_s\ V_{old}\ V_w$

$|H_1| = -1$ $\quad |H_2| = \begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix} = 4$ $\quad |H_3| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -8$

it means that f is a negative definite, so it is concave and local maximizers are also global solutions.

c) Consumer surplus and Producer surplus.

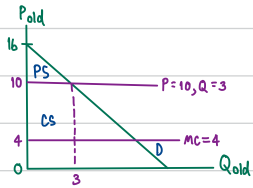
- Demand of student : $P_s = 8 - \frac{1}{2}Q_s$



Consumer surplus = $\frac{1}{2} \times (8-6) \times (4-0) = \frac{1}{2} \times 2 \times 4 = \$4$

Producer surplus = $(6-4) \times (4) = 2 \times 4 = \$8$

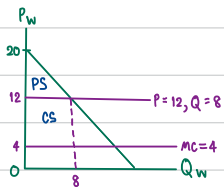
- Demand of senior citizens : $P_{old} = 16 - 2Q_{old}$



Consumer surplus = $\frac{1}{2} \times (16-10) \times (3-0) = \frac{1}{2} \times 6 \times 3 = \$9$

Producer surplus = $(10-4) \times (3) = 6 \times 3 = \$18$

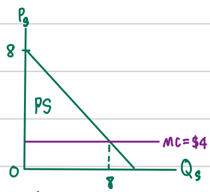
- Demand of working-aged people : $P_W = 20 - Q_W$



Consumer surplus = $\frac{1}{2} \times (20-12) \times (8-0) = \frac{1}{2} \times 8 \times 8 = \$32$

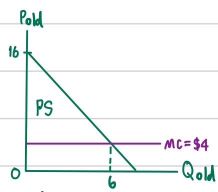
Producer surplus = $(12-4) \times (8) = 8 \times 8 = \$64$

d) Student



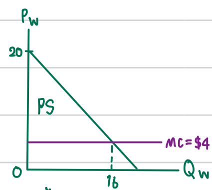
$Q^* = 8$ #

Senior citizens



$Q^* = 6$ #

Working-aged people



$Q^* = 16$ #

Question 4:

Suppose that the output Q of a firm depends on two inputs of the quantities K and L. The output level is determined by the production function

$$Q = 36K + 16L - 3K^2 - 2KL - L^2$$

a. Is the firm's production function strictly concave? Explain.

$$H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix}$$

$$\frac{\partial Q}{\partial K} = 36 - 6K - 2L$$

$$\frac{\partial Q}{\partial L} = 16 - 2K - 2L$$

$$H = \begin{bmatrix} -6 & -2 \\ -2 & -2 \end{bmatrix}$$

$$|H_1| = -6 \; ; \; \forall K, \forall L \longrightarrow |H_1| < 0$$

$$|H_2| = (-6)(-2) - (-2)(-2) = 8 \; ; \; \forall K, \forall L \longrightarrow |H_2| > 0$$

$\therefore$ For any value of K and L, $|H_1| < 0$ and $|H_2| > 0$. $d^2Q < 0$, the firm's production function is strictly concave.

b. Determine the optimal input (K*, L*) that maximizes the output level.

$$\frac{\partial Q}{\partial K} = 36 - 6K - 2L = 0$$

$$L = 18 - 3K \text{ — } ①$$

$$\frac{\partial Q}{\partial L} = 16 - 2K - 2L = 0$$

$$L = 8 - K \text{ — } ②$$

$$① = ②$$

$$18 - 3k = 8 - k$$

$$2k = 10$$

$$K^* = 5 \#$$

Plug $K^* = 5$ in ②

$$L^* = 8 - 5 = 3 \#$$

c. Write down the firm's profit function when the price of Q is P and the per-unit factor prices of K and L are r and w, respectively, where both r and w are positive numbers. Find the levels of K^* and L^* that maximize the firm's profits.

Profit Function = Total Revenue − Total Cost

$$\pi = P \cdot Q - rk - wL$$

$$\pi = P \cdot (36K + 16L - 3K^2 - 2KL - L^2) - rk - wL$$

Profit Maximizing input factors

$$\frac{\partial \pi}{\partial K} = 36P - 6KP - 2LP - r = 0$$

$$P(36 - 6K - 2L) = r$$

$$\frac{36 - 6K - \frac{r}{P}}{2} = L$$

$$L = 18 - 3k - \frac{r}{2P} \text{ — } ①$$

$$\frac{\partial \pi}{\partial L} = 16P - 2KP - 2LP - W = 0$$

$$16 - 2K - 2L = \frac{W}{P}$$

$$L = 8 - K - \frac{W}{2P} \text{ — } ②$$

$$① = ②$$

$$18 - 3k - \frac{r}{2P} = 8 - K - \frac{W}{2P}$$

$$10 - \frac{r}{2P} + \frac{W}{2P} = 2k$$

$$K^* = 5 - \frac{r}{4P} + \frac{W}{4P} \#$$

Plug K^* into ②

$$L = 8 - \left(5 - \frac{r}{4P} + \frac{W}{4P}\right) - \frac{W}{2P}$$

$$L^* = 3 + \frac{r}{4P} - \frac{W}{4P} - \frac{W}{2P}$$

$$L^* = 3 + \frac{r}{4P} - \frac{W}{4P} \#$$

d. Verify that the second-order sufficient conditions for maximum profits are satisfied.

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} -6P & -2P \\ -2P & -2P \end{bmatrix}$$

$|H_1| = -6P$; $\forall K, \forall L \longrightarrow |H_1| < 0$

$|H_2| = (-6P)(-2P) - (-2P)(-2P) = 8P^2$; $\forall K, \forall L \longrightarrow |H_2| > 0$

$\therefore$ For any value of K and L, $|H_1| < 0$ and $|H_2| > 0$. $d^2Q < 0$, K^* and L^* are the profit maximizers.

e. Determine the effect of an increase in r on the firm's use of each input. (i.e. determine $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial r}$).

$K^* = 5 - \frac{r}{4P} + \frac{W}{4P}$; $L^* = 3 + \frac{r}{4P} - \frac{W}{4P}$

$\frac{\partial K^*}{\partial r} = -\frac{1}{4P}$ #

When the cost of capital increases by 1 unit, the optimal K^* falls by $\frac{1}{4P}$ unit.

$\frac{\partial L^*}{\partial r} = \frac{1}{4P}$ #

When the cost of capital increases by 1 unit, the optimal L^* rises by $\frac{1}{4P}$ unit.