

# EE320 (2/2012)

## INTRODUCTORY MATHEMATICAL ECONOMICS

---

NONLINEAR MODEL AND DIFFERENTIAL CALCULUS  
IN ECONOMIC THEORY

(Part 2 – Optimum and Extreme Values)

# Topics

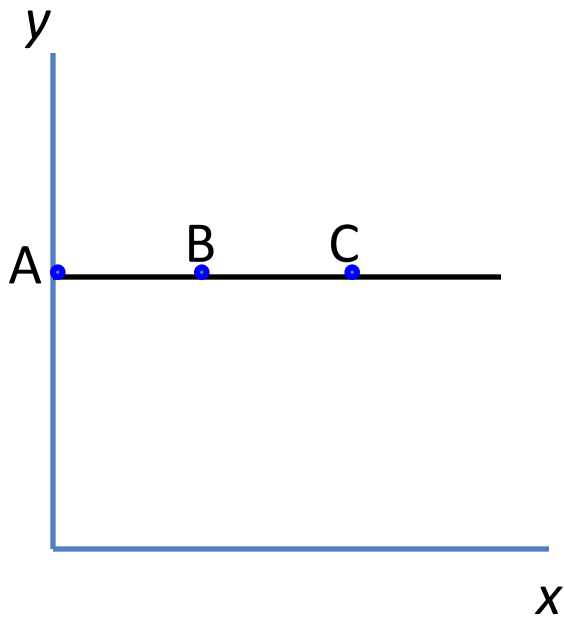
- Optimal Values and Extreme Values
- First-Derivative Test for Relative Extremum
- Second and Higher Derivatives
- Convexity and Concavity
- Second-Derivative Test for Relative Extremum

# Optimal Values and Extreme Values

- Most common criterion of choice among alternatives in economics is to find an *extreme* value – *maximum* or *minimum*.
- *Optimization* problems could be *maximizing* something (e.g. maximizing profit) or *minimizing* something (e.g. minimizing costs).
  - *Objective function* → dependent variable
  - *Choice* → independent variable
- Example: Firm's objective is to maximize profit:
$$\pi(Q) = R(Q) - C(Q).$$
  - $\pi$  is the object of maximization
  - $Q$  is the choice variable.
- This lecture will focus on the general objective function of one variable:  $y = f(x)$ .

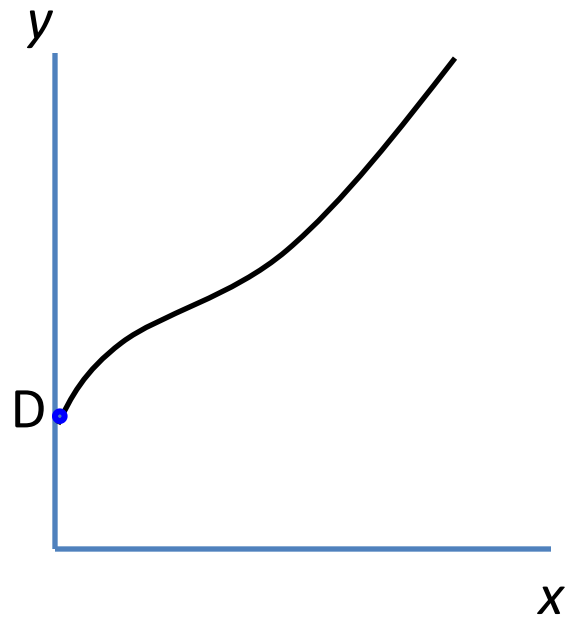
# Relative VS. Absolute Extremum

(a)



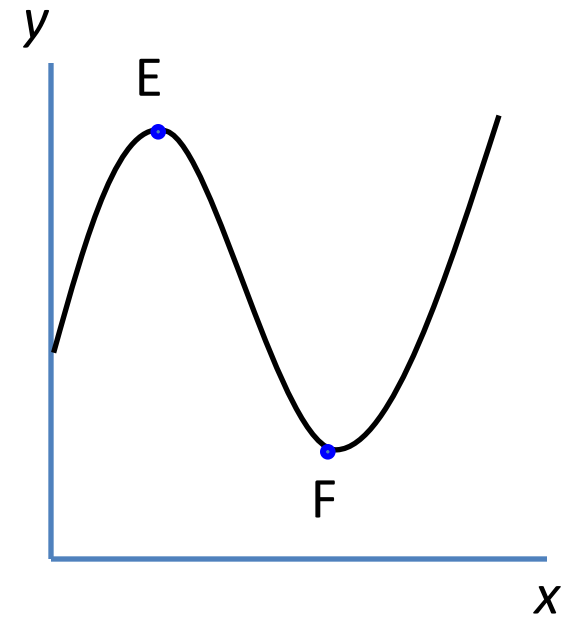
No extremum

(b)



Absolute (or global)  
minimum

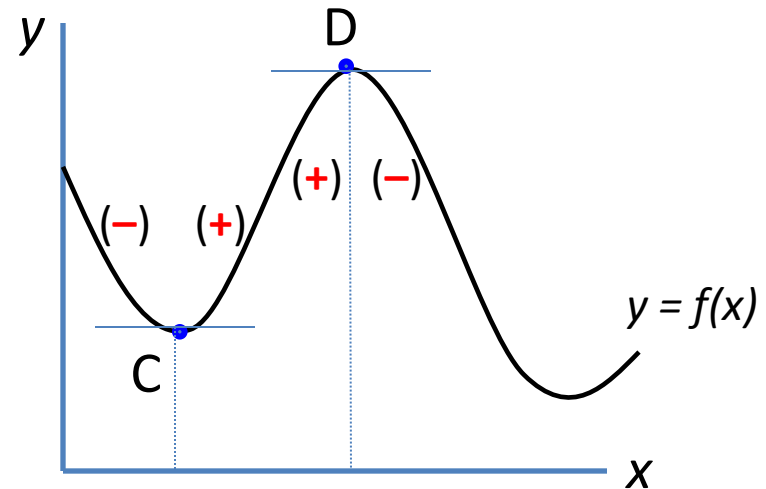
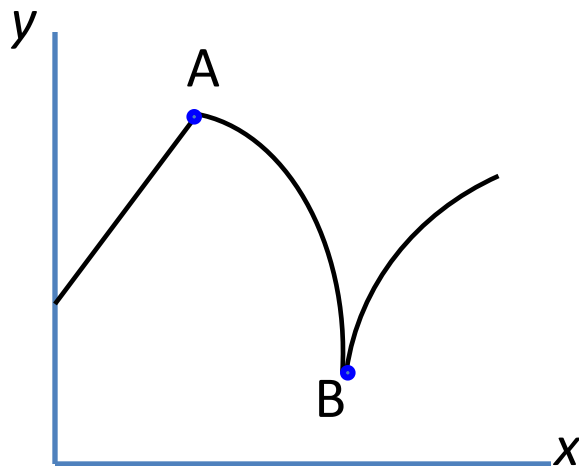
(c)



Relative (or local)  
extremums

# First-Derivative Test (1)

- 2 cases if a relative extremum of  $f(x)$  occurs at  $f(x_0)$ :
  1.  $f'(x_0)$  does not exist.
  2.  $f'(x_0) = 0$ .



For smooth functions, the *necessary* condition to determine relative extreme values (local max/local min) is  $f'(x) = 0$ .

# First-Derivative Test (2)

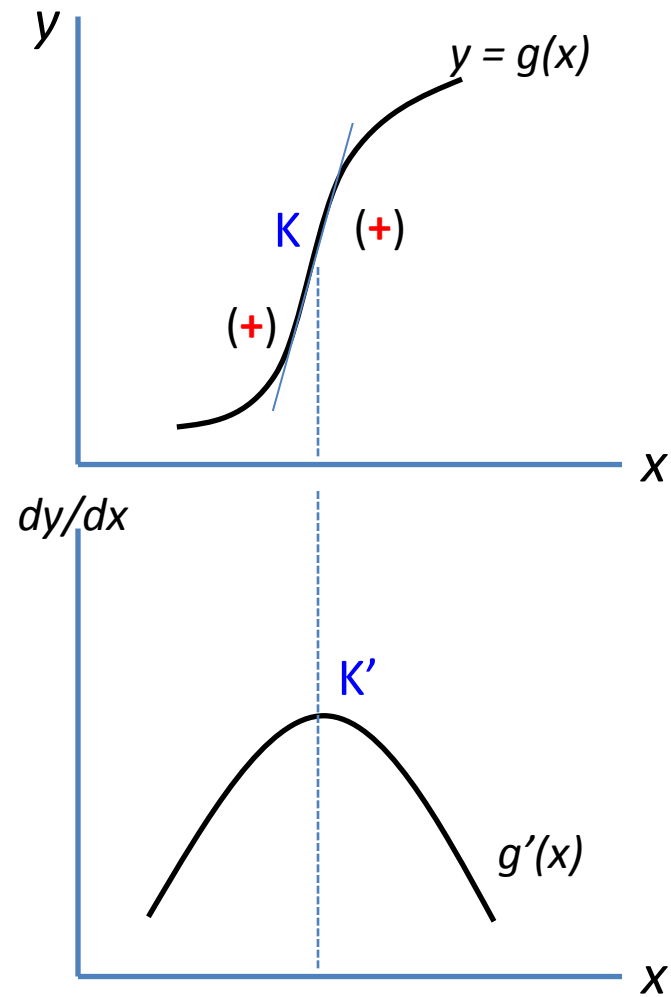
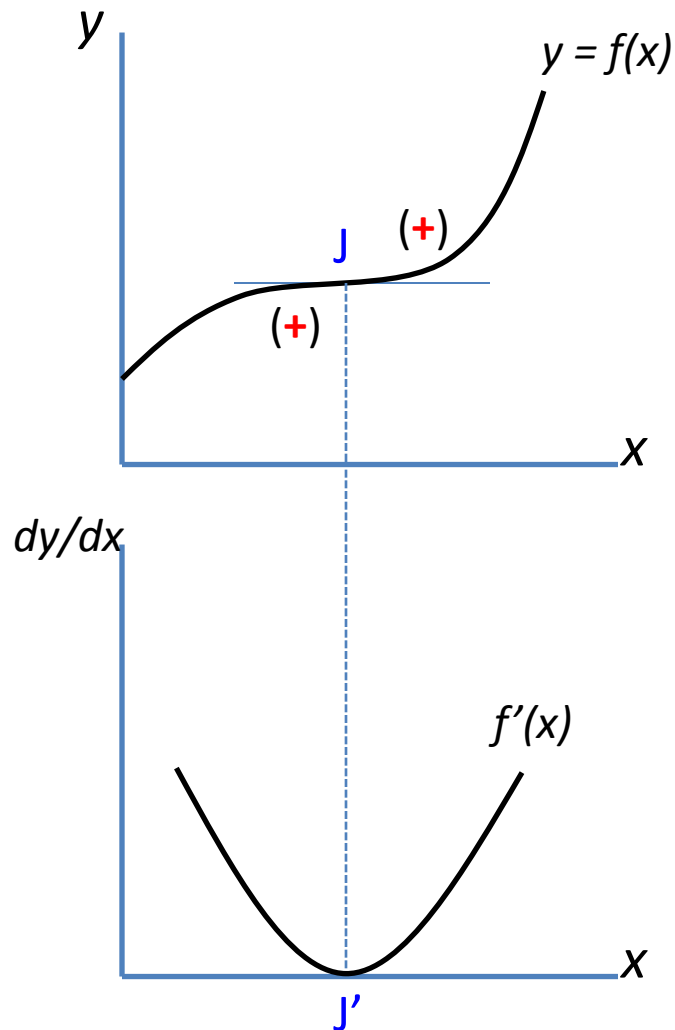
## First-Derivative Test for Relative Extremum:

If the first derivative of a function  $f(x)$  at  $x = x_0$  is  $f'(x_0) = 0$ , then the value of  $f(x_0)$  will be

- A relative *maximum* if  $f'(x)$  changes its sign from **+** to **-** from the immediate left of  $x_0$  to its immediate right.
- A relative *minimum* if  $f'(x)$  changes its sign from **-** to **+** from the immediate left of  $x_0$  to its immediate right.
- Neither* a relative maximum nor a relative minimum if  $f'(x)$  has **the same sign** on both the immediate left and the immediate right of  $x_0$ .

→  $x_0$  where  $f'(x_0) = 0$  is called the critical value, and  $f(x_0)$  is the stationary value of the function  $f(x)$ .

# Inflection Points



# Example

- Find the relative minimum of the average cost function

$$AC(Q) = Q^2 - 5Q + 8$$

→ Let  $f(Q) = AC(Q)$

$$f'(Q) = 2Q - 5$$

Necessary condition for an extremum:  $2Q - 5 = 0$

→  $Q^* = 2.5$

Check: if  $Q = 2$ ,  $f'(x) = -1$ ,

if  $Q = 3$ ,  $f'(x) = 1$ .

Thus,  $Q^* = 2.5$  is a minimum.

# Second and Higher Derivatives

- The derivative of a function  $f$  is called the **first derivative** of  $f$ .
- If  $f'$  is also differentiable, we can differentiate  $f'$  to get the **second derivative** of  $f$ :

- **Definition:**

$$f''(x) \equiv \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$$

- **Alternative notations:**  $\frac{d^2 f(x)}{dx^2}$ ,  $\frac{d^2 y}{dx^2}$ ,  $y''$

- As long as the differentiability condition is met, higher-order derivatives can be written similarly as:

$$f'''(x), f^{(4)}(x), \dots, f^{(n)}(x) \quad \text{or} \quad \frac{d^3 y}{dx^3}, \frac{d^4 y}{dx^4}, \dots, \frac{d^n y}{dx^n}$$

# Interpretation of the Second Derivative

- **First derivative:**

$f'(x_0) > 0$  : the *value of the function* tends to *increase*.

$f'(x_0) < 0$  : the *value of the function* tends to *decrease*.

- **Second derivative:**

$f''(x_0) > 0$  : the *slope of the curve* tends to *increase*.

$f''(x_0) < 0$  : the *slope of the curve* tends to *decrease*.

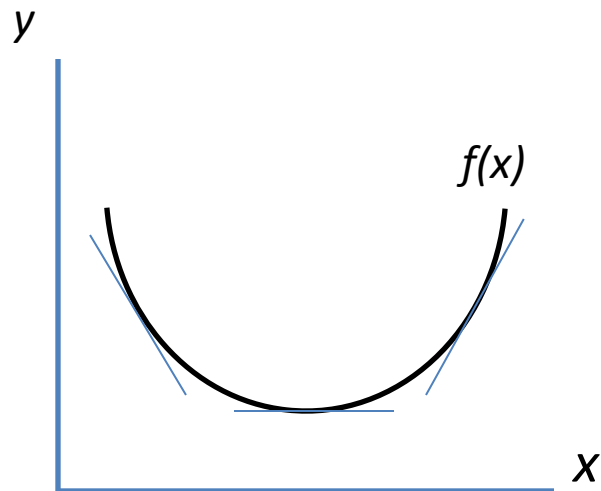
- **Possible cases:**

- $f'(x) > 0$  and  $f''(x) > 0$  : the slope is *positive and increasing* as x increases.
- $f'(x) < 0$  and  $f''(x) > 0$  : the slope is *negative and increasing* as x increases.
- $f'(x_0) > 0$  and  $f''(x) < 0$  : the slope is *positive and decreasing* as x increases.
- $f'(x_0) < 0$  and  $f''(x) < 0$  : the slope is *negative and decreasing* as x increases.

# Curvature of a Graph

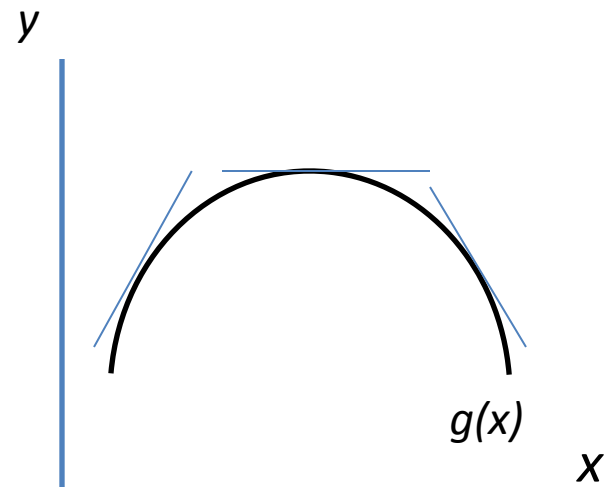
$$f''(x) > 0:$$

Slope of the tangent increases as  $x$  increases (i.e.  $f'(x)$  is increasing).



$$g''(x) < 0:$$

Slope of the tangent decreases as  $x$  increases (i.e.  $g'(x)$  is decreasing).



# Convexity and Concavity (1)

## Definition: Strictly Convex

A function  $f(x)$  is *strictly convex* if we pick any pair of points  $M$  and  $N$  on its curve and join them by a straight line, the line segment  $MN$  must lie entirely *above* the curve, excepts at points  $M$  and  $N$ .

## Definition: Strictly Concave

A function  $f(x)$  is *strictly concave* if we pick any pair of points  $M$  and  $N$  on its curve and join them by a straight line, the line segment  $MN$  must lie entirely *below* the curve, excepts at points  $M$  and  $N$ .

Note: When the segment  $MN$  *either* lies above (below) *or* lies along the curve, the function is *convex* (*concave*).

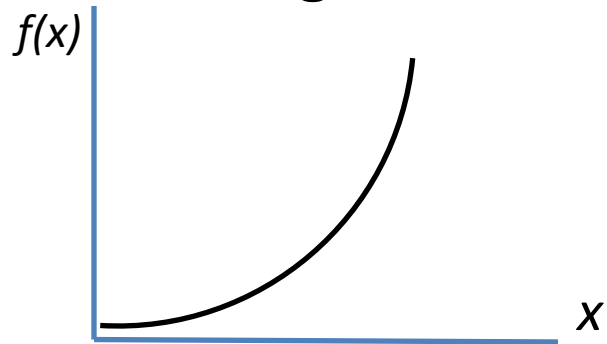
# Convexity and Concavity (2)

- **Definition:** Assume that  $f$  is continuous and twice differentiable.  
If  $f''(x) > 0$  for all  $x$ , then  $f(x)$  is a *strictly convex* function.  
If  $f''(x) < 0$  for all  $x$ , then  $f(x)$  is a *strictly concave* function.
- **Example 1:**  $f(x) = x^2 - 2x + 2$
- **Example 2:**  $f(x) = ax^2 + bx + c$

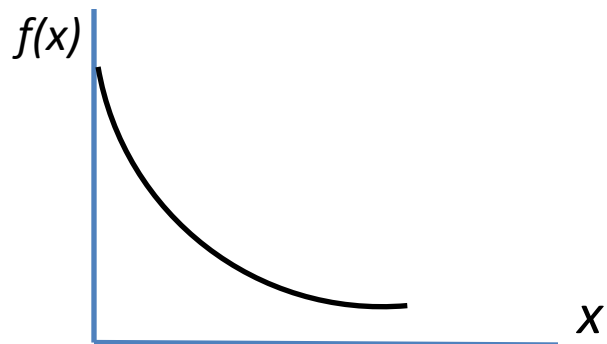
Note: In certain cases,  $f''(x)$  may have a zero value at the stationary point. Thus, the second derivative is a **sufficient but not necessary condition**. (Example:  $f(x) = x^4$ ).

# Convexity and Concavity (3)

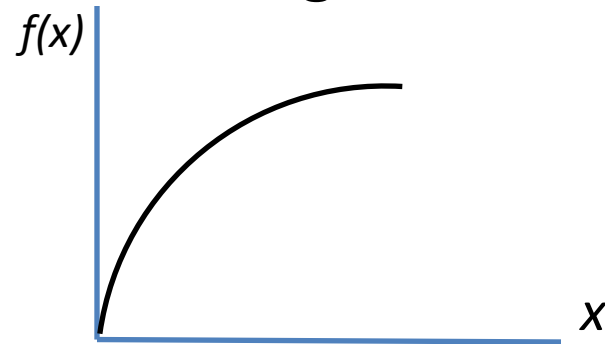
- Increasing Convex



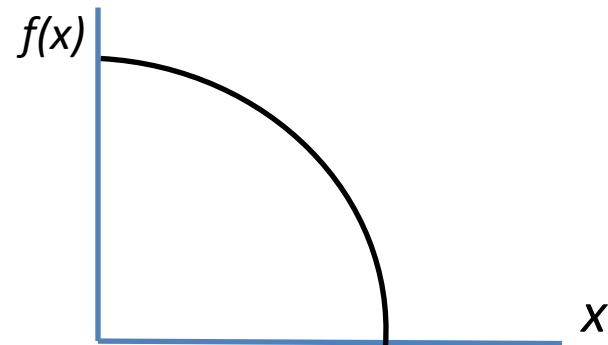
- Decreasing Convex



- Increasing Concave

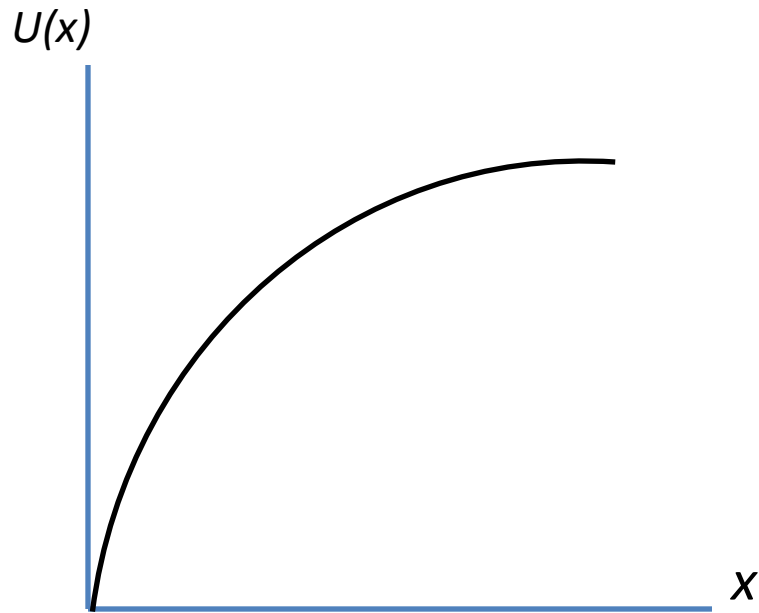


- Decreasing Concave

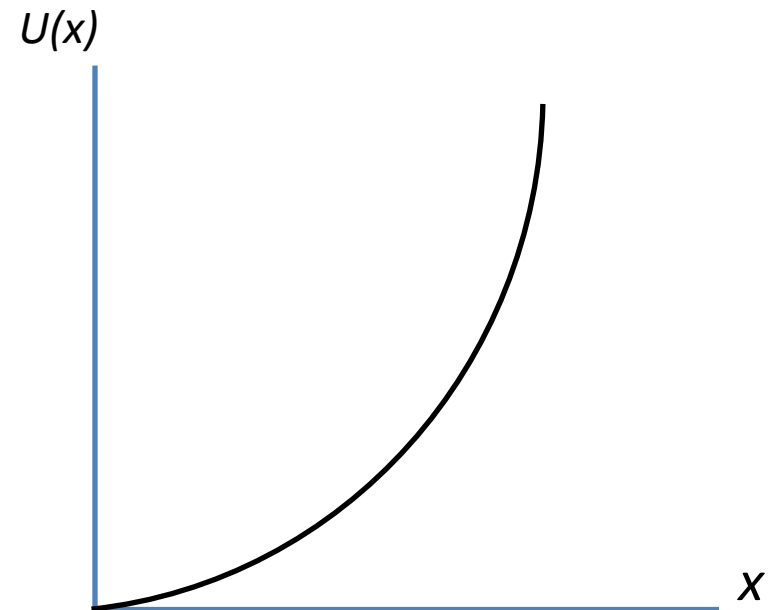


# Example: Attitudes toward Risk

## Risk-Averse Preference



## Risk-Loving Preference



# Second-Derivative Test

## Second-Derivative Test for Relative Extremum:

If  $f'(x_0) = 0$ , then the value of  $f(x_0)$  will be

- A relative *maximum* if  $f''(x) < 0$ .
- A relative *minimum* if  $f''(x) > 0$ .

**Example 1:** Determine the extremum of  $f(x) = x^3 - 3x^2 + 2$

**Example 2:** Determine the extremum of  $g(x) = \frac{x^4}{4} - \frac{3}{2}x^2$

# Necessary VS. Sufficient Conditions for Relative Extremum

Condition	Maximum	Minimum
First-order necessary	$f'(x) = 0$	$f'(x) = 0$
Second-order necessary	$f''(x) \leq 0$	$f''(x) \geq 0$
Second-order sufficient	$f''(x) < 0$	$f''(x) > 0$

Example: Determine the extremum of  $f(x) = x^4$

- FOC:  $f'(x) = 4x^3 = 0$  when  $x^* = 0$ .
- SOC:  $f''(x) = 12x^2 \geq 0$  is a second-order necessary condition for a minimum.

# Conditions for Profit Maximization

- Profit function:  $\pi(Q) = R(Q) - C(Q)$

- First-order necessary condition:

$$\pi'(Q) = R'(Q) - C'(Q) = 0$$

$$\Leftrightarrow MR(Q) = MC(Q)$$

(i.e. slope of revenue function = slope of cost function)

- Second-order sufficient condition:

$$\pi''(Q^*) < 0$$

$$\Leftrightarrow MR'(Q) = MC'(Q)$$

- Profit-maximizing output:

$Q^*$  such that  $\pi'(Q^*) = 0$ .

# Graph: Profit Maximization

# Example: Profit Maximization

Let  $R(Q) = 4000Q - 33Q^2$

$$C(Q) = 2Q^3 - 3Q^2 + 400Q + 5000$$

Determine the profit maximizing output level.

FONC:  $\pi'(Q) = R'(Q) - C'(Q) = 0$

$$\Leftrightarrow (4000Q - 33Q^2) - (2Q^3 - 3Q^2 + 400Q + 5000) = 0$$

$$\Leftrightarrow Q^* = 20$$

SOSC:  $\pi''(20) = -300 < 0$ .

Thus,  $Q^* = 20$  is the profit-maximizing output level.