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1. Introduction

1.1 Review of Some Statistical Concepts

1.1.1 Probability and Random Variable

Probability is the possibility that any event will occur, given some specific sample space.

Let A be the event occurring in the given sample space and $P(A)$ be the probability that A will happen. Then, $P(A)$ is defined as;

$$P(A) = \frac{\text{the number of times the event A will occur}}{\text{the number of all possible outcomes in sample space}} \tag{1.1}$$

For instance, to draw one card from the standard 52-card deck, let A be the event that the rank of card is 2. Times the event will occur is 4 and the amount of all possible outcomes is 52; hence, the probability of A is $\frac{4}{52}$ or $\frac{1}{13}$.

Some properties of probability are;

1. $0 \leq P(A) \leq 1$

2. If A, B and C are exhaustive set, then,

$P(A) + P(B) + P(C) = 1$

3. If A, B and C are mutually exclusive, then,

$$P(A+B+C) = P(A) + P(B) + P(C)$$

Suppose that the results of an experiment are in the form of value, the variable, whose value is determined by one of those results, is known as **Random Variable**. Random variable can be either **discrete** or **continuous value**.

For discrete random variable, the example is the sum of the values on the face of two dice, when rolling two dice once. In other word, the obtained sum will range from 2 to 12, and it is impossible to get 2.5 or 3.5.

For continuous random variable, the example is the height of the high-school student, constricted to the range from 160 to 180 centimetres. It can be seen that the value of the height need not be the integers and can take the value of 160.5 or 160.52 centimetres.

These two distinct characteristics of random variable enable us to classify them into different probability density functions, which would be stated in section 1.4.

1.2 Probability Density Function (PDF)

As the value of random variable depends on an experiment, the **probability density function** would portray the overall image of possible random results. The type of the probability density function relies on the characteristics of the random variable. In this section, many important types are discussed.

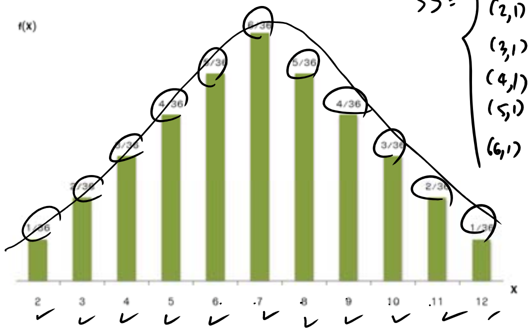
1.2.1 Probability Density Function for Discrete Random Variable

Let X be the discrete random variable with the value $x_1, x_2, \ldots, x_n$ and we get,

$$\boxed{\begin{aligned} f(x) &= P(X = x_i) \\ f(x) &= 0 \end{aligned}} \quad \begin{aligned} &\text{for} \quad i = 1, 2, \ldots, n \\ &\text{for} \quad x \neq x_i \end{aligned}$$

Example: Let X be random variable of the sum of values on the face of two dices. The value might be 2 or 12, that is the value from both rolling round is 1 or 6, respectively. The Figure 1 summarizes all possible results#

Figure 1: Probability Density function of the Sum of Values on the Side of the Dice, Obtained from Rolling the Dice Twice



$SS = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ (2,1), (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6) \\ (3,1) \\ (4,1) \\ (5,1) \\ (6,1)\ (6,2)\ (6,3), (6,4), (6,5)(6,6) \end{Bmatrix}$ ← 36 possible outcomes

$P(X=2) = \frac{1}{36}$

$P(X=7) = \frac{6}{36}$

$E(X) = \left(\frac{1}{36}\right)\cdot 2 + \left(\frac{2}{36}\right)\cdot 3 + \left(\frac{3}{36}\right)\cdot 4 + \left(\frac{4}{36}\right)\cdot 5 + \left(\frac{5}{36}\right)\cdot 6$
$+ \left(\frac{6}{36}\right)\cdot 7 + \left(\frac{5}{36}\right)\cdot 8 + \left(\frac{4}{36}\right)\cdot 9 + \left(\frac{3}{36}\right)\cdot 10 + \left(\frac{2}{36}\right)\cdot 11 + \left(\frac{1}{36}\right)\cdot 12$

$= \quad 7.$

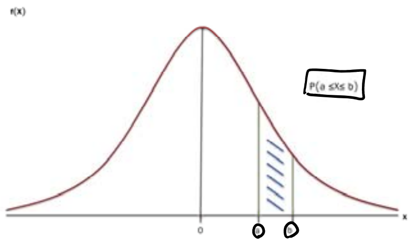
1.2.2 Probability Density Function for Continuous Random Variable

Let X be the continuous random variable. The probability density function of X satisfies the three following conditions.

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$ ✓
3. $\int_a^b f(x)dx = P(a \leq x \leq b)$

Figure 2 exhibits the probability density function for the continuous random variable, where the area under the curve represents the probability that the variable will lay on that range. Specifically, $P(a \leq X \leq b)$ means the probability that X will take the value between a and b.

Figure 2: Probability Density Function for Continuous Random Variable



Ex: $f(x) = \frac{1}{9}x^2 \quad 0 \leq x \leq 3$

$$\int_0^3 \frac{1}{9}x^2 \, dx = 1 \quad \rightarrow \text{Property \# 2}$$

let's see: $\int_0^3 \frac{1}{9}x^2 dx = \frac{1}{9}\int_0^3 x^2 dx = \frac{1}{9}\frac{x^{2+1}}{2+1} = \frac{1}{9}\frac{x^3}{3} = \frac{1}{27}x^3 \Big|_0^3 = \frac{1}{27}(3)^3 - \frac{1}{27}(0)^3 = \frac{27}{27} - 0 = 1 \ \#$

Ex: $\int_0^1 \frac{1}{9}x^2 dx = \frac{1}{27}x^3 \Big|_0^1 = \frac{1}{27}(1)^3 - \frac{1}{27}(0)^3 = \frac{1}{27}.$

1.2.3 Joint Probability Density Function

In this section, only **joint probability density function** for discrete variable is discussed. Let X and Y be discrete random variables. The joint probability density function, identifying the probability that X and Y happen simultaneously, is written as,

$$\boxed{f(X,Y) = P(X = x \text{ and } Y = y)}$$

$$\boxed{f(X = x,\ Y = y)}$$

Example: The following table explains the joint probability density function.

Table 1: The table illustrating the joint probability density function of X and Y

		X		
		-1	0	1
Y	1	0.11	0.08	0.05
	2	0.09	0.05	0.03
	3	0.35	0.07	0.17

$f(X=-1, Y=1) = 0.11 \quad f(X=0, Y=1) = 0.08 \quad f(X=1, Y=1) = 0.05$

$f(X=-1, Y=2) = 0.09 \quad f(X=0, Y=2) = 0.05 \quad f(X=1, Y=2) = 0.03$

$f(X=-1, Y=3) = 0.35 \quad f(X=0, Y=3) = 0.07 \quad f(X=1, Y=3) = 0.17$

		X		
		-1	0	1
Y	1	0.11	0.08	0.05
	2	0.09	0.05	0.03
	3	0.35	0.07	0.17

According to the table 1, the probability that random variable X will be 0 and random variable Y will be 3 is 0.07 or 7 percent. In mathematical term, it can be written as $f(X=0, Y=3) = 0.07$.

$f(X=-1, Y=2) = 0.09$ $f(X=0, Y=2) = 0.05$ $f(X=1, Y=2) = 0.03$

$f(X=-1, Y=3) = 0.35$ $f(X=0, Y=3) = 0.07$ $f(X=1, Y=3) = 0.17$

$\boxed{\text{Joint PDF}}$

1.2.4 Marginal Probability Density Function

The above joint probability density function $f(X,Y)$ shows the joint distribution of two variables. On the other hand, **marginal probability density function** with respect to joint probability function, displays the probability density function of single variable like $f(X)$, $f(Y)$, which can be derived from;

$$f(X) = \sum_Y f(X,Y) \quad \text{called} \quad \text{marginal PDF of X}$$
$$f(Y) = \sum_X f(X,Y) \quad \text{called} \quad \text{marginal PDF of Y}$$

where $\sum_Y$ or $\sum_X$ means the summation of probability over all values of X and Y respectively.

Example: According to Table 2 above, marginal PDF of X is obtained from

$$
\begin{aligned}
(X=-1) &= \Sigma_Y (f(x=-1, Y) \\
&= f(x=-1, Y=1) + f(x=-1, Y=2) + f(x=-1, Y=3) \\
&= 0.11 + 0.05 + 0.35 \\
&= 0.55 \\
f(X=0) &= \sum_Y f(X=0,Y) \\
&= f(X=0,Y=1)+f(X=0,Y=2)+f(X=0,Y=3) \\
&= 0.08+0.05+0.07 \\
&= 0.20 \\
f(X=1) &= \sum_Y f(X=1,Y) \\
&= f(X=1,Y=1)+f(X=1,Y=2)+f(X=1,Y=3) \\
&= 0.05+0.03+0.17 \\
&= 0.25
\end{aligned}
$$

X

Y	-1	0	1	
1	0.11	0.08	0.05	→ f(Y=1) = ?
2	0.09	0.05	0.03	→ f(Y=2) = ?
3	0.35	0.07	0.17	→ f(Y=3) = ?
	↓ f(X=-1) = 0.55	↓ f(X=0) = 0.20	↓ f(X=1) = 0.25	

$f(X=-1, Y=1) = 0.11$ $\quad$ $f(X=-1, Y=2) = 0.09$ $\quad$ $f(X=-1, Y=3) = 0.35$

Marginal PDF of Y is obtained from

$$
\begin{aligned}
f(Y=1) &= \Sigma_X (X, Y=1) \\
&= f(X=-1, Y=1) + f(X=0, Y=1) + f(X=1, Y=1) \\
&= 0.11 + 0.08 + 0.05 \\
&= 0.24 \\
f(Y=2) &= \sum_X f(X, Y=2) \\
&= f(X=-1, Y=2) + f(X=0, Y=2) + f(X=1, Y=2) \\
&= 0.09 + 0.05 + 0.03 \\
&= 0.17 \\
f(Y=3) &= \sum_X f(X, Y=3) \\
&= f(X=-1, Y=3) + f(X=0, Y=3) + f(X=1, Y=3) \\
&= 0.35 + 0.07 + 0.17 \\
&= 0.59
\end{aligned}
$$

According to the calculation above, the result can be summarized into Table 2.

Table 2 shows joint probability of random variable X and Y

		X			
		-1	0	1	
Y	1	0.11	0.08	0.05	$f(Y=1)$ $= 0.24$
	2	0.09	0.05	0.03	$f(Y=2)$ $= 0.17$
	3	0.35	0.07	0.17	$f(Y=3)$ $= 0.59$
		$f(X=-1)$ $= 0.55$	$f(X=0)$ $= 0.20$	$f(X=1)$ $= 0.25$	$\sum f(X) = 1$ $\sum f(Y) = 1$

1.2.5 Conditional Probability Density Function

Conditional probability density function is the probability of one event given that some events have already occurred. The function is written as,

$$f(X|Y) = P(X = x|Y = y)$$

This function can be obtained from the joint probability density function through,

Recipe → $f(X|Y) = \dfrac{f(X,Y)}{f(Y)}$ → JOINT PDF → MARGINAL PDF(of Y)

Example: According to Table 2, find $f(X=1|Y=2)$ and $f(Y=2|X=0)$

- $f(X=1|Y=2) = \dfrac{f(X=1, Y=2)}{f(Y=2)} = \dfrac{0.03}{0.17} = 0.176$
- $f(Y=2|X=0) = \dfrac{f(X=0, Y=2)}{f(X=0)} = \dfrac{0.05}{0.20} = 0.25$

$f(X=x_i, Y=y_i)$ → joint PDF

Y \ X	x_1	x_2	x_3	Marginal PDF of Y
y_1	$f(x_1,y_1)$	$f(x_2,y_1)$	$f(x_3,y_1)$	→ $f(Y=y_1) = \sum_{\text{all } x} f(x,Y) = f(Y=y_1, X=x_1) + f(Y=y_1, X=x_2) + f(Y=y_1, X=x_3)$
y_2	$f(x_1,y_2)$	$f(x_2,y_2)$	$f(x_3,y_2)$	→ $f(Y=y_2) = \sum_{\text{all } x} f(x,Y) = f(Y=y_2, X=x_1) + f(Y=y_2, X=x_2) + f(Y=y_2, X=x_3)$
y_3	$f(x_1,y_3)$	$f(x_2,y_3)$	$f(x_3,y_3)$	→ $f(Y=y_3) = \sum_{\text{all } x} f(x,Y) = f(Y=y_3, X=x_1) + f(Y=y_3, X=x_2) + f(Y=y_3, X=x_3)$
Marginal PDF of X	$f(X=x_1) = \sum_{\text{all } y} f(x,Y)$	$f(X=x_2) = \sum_{\text{all } y} f(x,Y)$	$f(X=x_3) = \sum_{\text{all } y} f(x,Y)$	

$= f(X=x_1, Y=y_1) + f(X=x_1, Y=y_2) + f(X=x_1, Y=y_3)$

- $f(X=x_i, Y=y_i)$ → JOINT PDF ✓
- $f(X=x_i)$ → MARGINAL PDF OF X ✓
- $f(Y=y_i)$ → marginal PDF of Y ✓
- $f(X=x_i | Y=y_i)$ → "Conditional PDF" ✓

given that $Y=y_i$ already occured

$(X=0|Y=1) =$
$=$
$=$
$=$
$(Y=2|X=0) =$
$=$
$=$

1, 3, 5

Example: Let event A be tossing the dice once and the point is odd number and B be the tossing the dice once and the point is at least 5. Find the probability that the point coming up is odd given that the point has to be at least 5. = P(A|B)

5, 6

Answer A and B will occur simultaneously if the point from tossing the dice is 5; so, the joint probability of A and B is $\frac{1}{6}$. The probability that B occurs is $\frac{2}{6}$. Hence, the conditional probability of A given B is

$$\boxed{P(A|B) = \frac{P(A \text{ and } B)}{P(B)}} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}\#$$

1.2.6 Statistical Independence

Two random variables are **independent** if the resulting value of one variable does not affect the resulting value of the other; namely,

JOINT PDF MARGINAL PDF of X · MARGINAL PDF OF Y

$$f(X,Y) = f(X)f(Y)$$

Example: Consider Mr. Ake's expenditure for a meal and the Miss Somsri's expenditure for a dessert. Given that they do not know each other, the realization of Mr. Ake's expenditure does not imply the realization of Miss Somsri's expenditure. We can, thus, conclude that the expenditures of these two people are independent#

Example: Consider drawing cards sequentially from the standard 52-card deck without putting it back into the deck. Once the first card is drawn, the probability of drawing the second card will be influenced because the amount of cards in the deck is reduced. In this case, it can be concluded that drawing the first and second card are not independent#

1.3 Expectation, Variance, Covariance and Correlation

1.3.1 Mean or Expected Value

Because the value of random variable hinges on the value of random results of experiment which cannot be determined certainly, statisticians have invented the measures of central tendency of the random variable. One of them is **expected value**, indicating the mean of the random variable.

For discrete random variable, the expected value is calculated by;

$$\boxed{E(X) = \sum_{i=1}^{n} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + \ldots + x_n f(x_n)}$$

For continuous random variable, the expected value is calculated by,

$$\boxed{E(X) = \int_a^b x f(x) dx}$$

where;

$E(X)$ is the measure of central tendency of random variable, resulting from repeated trial of experiment.

$\sum_{i=1}^{n} x_i f(x_i)$ is the average of random variable weighted by the probability corresponding to each value.

a and b are the lowest and highest constant possible respectively.

Example: Find the expected value of rolling two dice once (Figure 1)

(page 5)

$E(X) = 7.$

Crucial properties of expected value include:

1. $E(b) = b$
2. $E(aX+b) = aE(X)+b$
3. $E(XY) = E(X)E(Y)$; given that X and Y are independent
4. $E(g(X)) = \sum_x g(X)f(X)$

where a and b are constant.

Conditional **expectation value** is the expectation value of random variable under some conditions such as expected value of X conditional on Y or $E(X|Y=5)$

Let $f(X,Y)$ be the joint probability function of X and Y. The expectation of X conditional on some value of Y is defined as,

For discrete random variable $\boxed{E(X|Y=y) = \sum_X X_i f(X|Y=y)}$

For continuous random variable $E(X|Y=y) = \int_{-\infty}^{\infty} X_i f(X|Y=y)$

Random variable

		X: -1	0	1	
Random variable Y	1	0.11	0.08	0.05	$\to f(Y=1) = 0.24$
	2	0.09	0.05	0.03	$\to f(Y=2) = 0.17$
	3	0.35	0.07	0.17	$\to f(Y=3) = 0.59$
		$f(X=-1) = 0.55$	$f(X=0) = 0.20$	$f(X=1) = 0.25$	

$\sum_Y f(Y) = 1$

$\sum_X f(X) = 1$

- JOINT PDF
- MARGINAL PDF OF X
- MARGINAL PDF OF Y

Example $E(X|Y=1)$

$$= -1 \cdot f(X=-1|Y=1) + 0 \cdot f(X=0|Y=1) + 1 \cdot f(X=1|Y=1)$$

$$= -1 \cdot \frac{f(X=-1,Y=1)}{f(Y=1)} + 0 \cdot \frac{f(X=0,Y=1)}{f(Y=1)} + 1 \cdot \frac{f(X=1,Y=1)}{f(Y=1)}$$

$$= -1 \cdot \frac{0.11}{0.24} + 0 \cdot \frac{0.08}{0.24} + 1 \cdot \frac{0.05}{0.24}$$

$$= -0.25 \ \#$$

Given that Y = 1 has already occured, on average, X will be equal to -0.25.

1.3.2 Variance

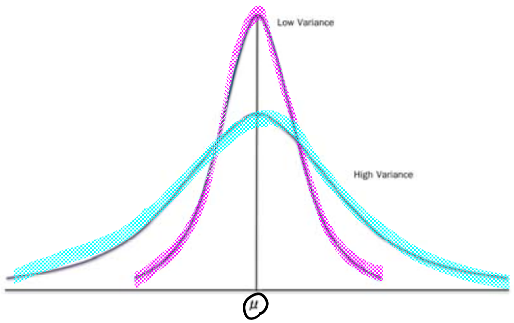
Variance is the measure of dispersion of the value of variable around the expected value. The higher the variance, the more dispersing the random variable (Figure 3). If X is the random variable with expected value μ, we get;

$$\boxed{Var(X) = \sigma_X^2 = E[X - E(X)]^2 = E(X)^2 - \mu^2} \tag{1.2}$$

From,

$$\left.\begin{aligned} Var(X) &= \sigma_X^2 \\ &= E[X - E(X)]^2 \\ &= E[X^2 - 2XE(X) + (E(X))^2] \\ &= E(X^2) - 2(E(X))^2 + (E(X))^2 \\ &= E(X)^2 - \mu^2 \end{aligned}\right\} \text{Proof}$$

Figure 3: Distribution of Random Variables with Different Variance



Important properties of expected value include;

1. $Var(b) = 0$
2. $Var(aX + b) = a^2 Var(X)$
3. $Var(X \pm Y) = Var(X) + Var(Y)$; given that X and Y are independent
4. $Var(aX \pm bY) = a^2 Var(X) + b^2 Var(Y)$

where a and b are constant.

1.3.3 Conditional Variance

The conditional variance of X is given Y = y is defined as following:

$$\begin{aligned} var(X|Y=y) &= E\{[X - E(X|Y=y)]^2 | Y=y\} \\ &= \sum_x [X - E(X|Y=y)]^2 f(x|Y=y) \\ &= \int_{-\infty}^{\infty} [X - E(X|Y=y)]^2 f(x|Y=y) dx \end{aligned} \tag{1.3}$$

→ Recipe to compute....

Example

$$VAR(X \mid Y=1) = \sum_x [x - E(X \mid Y=1)]^2 \cdot f(X \mid Y=1)$$

$E(X|Y=1) = -0.25$ (from page 14)

$$Var(X|Y=1) = [-1 - (-0.25)]^2 \cdot \frac{\overset{0.11}{f(x=-1, Y=1)}}{\underset{0.24}{f(Y=1)}} + [0 - (-0.25)]^2 \cdot \frac{\overset{0.08}{f(x=0, Y=1)}}{\underset{0.24}{f(Y=1)}} + [1 - (-0.25)]^2 \frac{\overset{0.05}{f(x=1, y=1)}}{\underset{0.24}{f(Y=1)}}$$

$$= 0.6042$$

Properties of conditional expectation and conditional variance

① If $f(x)$ is a function of x, then $E[f(x)|x] = f(x)$

Ex: $E[x^3|x] = E[x^3] = x^3$.

② If $f(x)$ and $g(x)$ are function of x, then

$$E[f(x)\cdot Y + g(x) | x] = f(x)\cdot E[Y|x] + E[g(x)|x]$$
$$= f(x)\cdot E[Y|x] + g(x)$$

use property #1

Ex: $E[XY + cX^2 | X] = X\cdot E[Y|X] + c\cdot E[X^2|X]$
$$= X\cdot E[Y|X] + c\cdot X^2$$

again, use property #1

where c is a constant term.

③ If X and Y are independent random variables, then $\underbrace{E[Y|X]}_{\text{conditional expected value of } Y} = \underbrace{E[Y]}_{\text{unconditional expected value of } Y}$.

$E[X|Y] = E[X]$, likewise.

④ The Law of Iterated Expectation: $E[Y] = E_X[E[Y|X]]$

it said... unconditional expectation is equal to ... Expectation of its conditional expectation.

Ex: If $E[Y|X]=0$, then $E[E[Y|X]] = ?$

Answer: $E[E[Y|X]] = E[0] = 0$.

⑤ If X and Y are "independent," then $var[Y|X] = var[Y]$!!!

⑥ $$var[Y] = \underbrace{E[var(Y|X)]}_{\text{Expected value of Conditional variance of } Y} + \underbrace{var[E(Y|X)]}_{\text{variance of its' conditional expected value of } Y}$$

1.3.4 Covariance

Theorem. Let X and Y be two random variables with means μ_x and μ_y, respectively. Then, we can define the covariance between these two variables as following:

$$cov(X,Y) = E\{(\underbrace{X-\mu_x}_{\text{MEAN OF X}})(\underbrace{Y-\mu_y}_{\text{MEAN OF Y}})\} = E(XY) - \mu_x\mu_y \quad (1.4)$$

Proof: $E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y]$
$$= E[XY] - E[X]\mu_Y - E[Y]\mu_X + E[\mu_X\mu_Y]$$
$$= E[XY] - \cancel{\mu_X\mu_Y} - \mu_Y\mu_X + \cancel{\mu_X\mu_Y}$$
$$= E[XY] - \mu_X\mu_Y.$$

Note: $E(X) = \mu_X$
$E(Y) = \mu_Y$.
$E[\mu_X\cdot\mu_Y] = \mu_X\cdot\mu_Y$

If X and Y are continuous random variables we can calculate their cov(X,Y):

$$cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X-\mu_x)(Y-\mu_y)f(x,y)dxdy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}XYf(x,y)dxdy - \mu_x\mu_y \quad (1.5)$$

Properties of Covariance

$$
\boxed{\begin{aligned} cov(X,Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X-\mu_x)(Y-\mu_y)f(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} XYf(x,y)dxdy - \mu_x\mu_y \end{aligned}} \quad (1.5)
$$

$E[\mu_x \cdot \mu_y] = \mu_x \cdot \mu_Y$

Properties of Covariance

1. If X and Y are independent, the covariance between X and Y is zero.

Proof: First, note that when x and Y are independent, $E[XY] = E[X] \cdot E[Y]$

$= \mu_x \cdot \mu_Y$. #

$$
\begin{aligned}
Cov(x,Y) &= E[XY] - \mu_x\mu_Y \\
&= E[X] \cdot E[Y] - \mu_x\mu_Y \\
&= \mu_x \cdot \mu_Y - \mu_x\mu_Y \\
&= 0.
\end{aligned}
$$

Recall: $cov(x,Y) = E[(x-\mu_x)(Y-\mu_y)]$

2. $cov(a+bX, c+dY) = bd * cov(X,Y)$, where a,b,c, and d are constants.

Proof:

$$
\begin{aligned}
Cov(a+bx, c+dY) &= E\{[(a+bx) - E(a+bx)][(c+dY) - E(c+dY)]\} \\
&= E\{[\cancel{a}+bx - \cancel{a} - bE(x)][\cancel{c}+dY - \cancel{c} - dE(Y)]\} \\
&= E\{[bx - bE(x)][dY - dE(Y)]\} \\
&= E\{b[x - E(x)] \cdot d[Y - E(Y)]\} \\
&= b \cdot d \, E\{[x - E(x)][Y - E(Y)]\} \\
&= b \cdot d \cdot cov(x,Y) \quad \#
\end{aligned}
$$

Example Suppose the join PDF of random variables X and Y can be represented as in the below table. What is the covariance between X and Y?

		X 1	X 2	X 3	
Y	1	0.25	0.25	0	f(Y = 1) = 0.5 = $\frac{1}{2}$
Y	2	0	0.25	0.25	f(Y = 2) = 0.5 = $\frac{1}{2}$
		f(X = 1) = 0.25 OR = $\frac{1}{4}$	f(X = 2) = 0.50 OR = $\frac{1}{2}$	f(X = 3) = 0.25 OR = $\frac{1}{4}$	f(X) = f(Y) =

$f(X=x_i, Y=y_i)$ JOINT PDF

$f(X=x_i)$ Marginal PDF

$f(Y=y_i)$ Marginal PDF

$$Cov(X,Y) = E[XY] - \mu_X \mu_Y$$

① Find E(X)

$$E(X) = \sum_x x \cdot f(x) = 1\cdot f(X=1) + 2\cdot f(X=2) + 3 \cdot f(X=3)$$
$$= 1\cdot(0.25) + 2\cdot(0.50) + 3\cdot(0.25)$$
$$= 2.00$$

② Find E(Y)

$$E(Y) = \sum_y y \cdot f(y) = 1\cdot f(Y=1) + 2\cdot f(Y=2)$$
$$= 1\cdot(0.50) + 2\,(0.50)$$
$$= 1.50$$

③ Find E[XY]

$$E[XY] = \sum_y \sum_x x\cdot y\ f(X=x, Y=y)$$

		X 1	X 2	X 3
Y	1	0.25	0.25	0
Y	2	0	0.25	0.25

$$= 1\cdot1\cdot f(X=1,Y=1) + 2\cdot1\cdot f(X=2,Y=1) + 1\cdot3\cdot f(X=3,Y=1)$$
$$+ 1\cdot2\cdot f(X=1,Y=2) + 2\cdot2\cdot f(X=2,Y=2) + 3\cdot2\cdot f(X=3,Y=2)$$
$$= 1\cdot1\cdot0.25 + 2\cdot1\cdot0.25 + 3\cdot2\cdot0$$
$$+ 1\cdot2\cdot0 + 2\cdot2\cdot0.25 + 3\cdot2\cdot0.25$$
$$= 0.25 + 0.5 + 0 + 0 + 1 + 1.5 = 3.25$$

$$\boxed{\begin{aligned} Cov(X,Y) &= E[XY] - \mu_X \cdot \mu_Y \\ &= 3.25 - 2\cdot1.5 \\ &= 0.25.\ \# \end{aligned}}$$

Next, we will turn our attention to seeing how we can apply the covariance to calculate the correlation between the random variables X and Y

1.3.5 Correlation

When we calculate the covariance of X and Y, it reflects the units of both random variables. However, it is useful to have **a dimensionless measure of dependency** by calculating the correlation instead.

Definition Let X and Y be any two random variables (discrete or continuous) with standard deviation σ_X and σ_Y, respectively. The **correlation coefficient** of X and Y, denoted **corr(X,Y)** or ρ_{XY} (the greek letter "rho") is defined as:

$$\rho_{XY} = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{cov(x,y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$\sigma_X \rightarrow$ Standard deviation of X

$\sigma_Y \rightarrow$ " ———— " of Y

greek letter "rho") is defined as:

$$\boxed{\rho_{XY} = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{cov(x,y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}}$$

$\sigma_X \to$ Standard deviation of X

$\sigma_Y \to$ " ——————— " of Y

Example Suppose the join PDF of random variables X and Y can be represented as in the below table. What is the correlation between X and Y?

		X: 1	2	3	
	1	0.25	0.25	0	$f(Y=1)$ =0.5
Y	2	0	0.25	0.25	$f(Y=2)$ =0.5
		$f(X=1)$ =0.25	$f(X=2)$ =0.5	$f(X=3)$ =0.25	$f(X)=1$ $f(Y)=1$

$$\rho_{XY} = \frac{cov(X,Y)}{\sqrt{var(X)\cdot var(Y)}} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}$$

$$var(X) = \sum (x-\mu_x)^2 \cdot f(x)$$

$$= (1-2)^2 \cdot \frac{1}{4} + (2-2)^2 \cdot \frac{1}{2} + (3-2)^2 \cdot \frac{1}{4}$$

$$= \frac{1}{4} + 0 + \frac{1}{4} = \frac{1}{2}.$$

$$var(Y) = \sum (Y-\mu_Y)^2 \cdot F(Y)$$

$$= (1-1.5)^2 \cdot \frac{1}{2} + (2-1.5)^2 \cdot \frac{1}{2}$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

$$CORR(X,Y) = \frac{\sigma_{XY}}{\sqrt{var(x)\cdot var(y)}} = \frac{0.25}{\sqrt{\frac{1}{2}\cdot\frac{1}{4}}} = 0.707$$

From the definition, ρ_{XY} is measure of linear association between two random variables. The value of ρ lies between -1 and +1, $-1 \leq \rho_{XY} \leq +1$. We can interpret the value of correlation as:

- If $\rho_{XY} = 1$, then X and Y are perfectly, positively, linearly correlated.
- If $\rho_{XY} = -1$, then X and Y are perfectly, negatively, linearly correlated.
- If $\rho_{XY} = 0$, then X and Y are completely,un-linearly correlated. This means that X and Y may correlated in some other manner i.e. a parabolic manner., but NOT in a linear manner
- If $\rho_{XY} \leq 0$, then X and Y are negatively linearly correlated, but NOT perfectly.
- If $\rho_{XY} \geq 0$, then X and Y are positively linearly correlated, but NOT perfectly.

Theorem. ① If X and Y are independent random variables, then:

① → ②

② $$corr(X,Y) = cov(X,Y) = 0$$

Example: Let X = the outcome of a fair, black, 6-sided die.
Let Y =outcome of a fair, red, 4-sided die.
What is the covariance of X and Y? What is the correlation of X and Y? ✓
=0 =0

		X: 1	2	3	4	5	6
	1	$\frac{1}{6}\cdot\frac{1}{4}=\frac{1}{24}$	·	·	·	·	$\frac{1}{24}$
Y	2	·	·	·	·	·	·
	3	·	·	·	·	·	·

$$COV[X,Y] = E[XY] - \mu_X \cdot \mu_Y$$

$\mu_X = ?$

$\mu_Y = ?$

$E[X\cdot Y] = ?$

1 $\frac{}{6}$ 4 $\frac{}{24}$ . . . $\frac{}{24}$

Y 2 . . . . . .

3 . . . . .

4 $\frac{1}{24}$ . . . . . $\frac{1}{24}$

$\mu_X = ?$

$\mu_Y = ?$

$E[X \cdot Y] = ?$

$\mu_X = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$

$= \frac{1}{6}(1+2+3+4+5+6)$

$= 3.5$

$\mu_Y = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4}$

$= \frac{1}{4}(1+2+3+4)$

$= 2.5$

$E[XY] = E[X] \cdot E[Y]$ since X & Y are independent!

$= 3.5 \cdot 2.5$

$= 8.75$

So $cov(X,Y) = E[XY] - \mu_X \mu_Y$

$= 8.75 - (3.5 \cdot 2.5)$

$= 8.75 - 8.75$

$= 0$

D-I-Y: verify that $corr(X,Y) = 0$ too!

NOTE: The converse of the theorem is NOT NECESSARILY CORRECT!

② → ① is not necessarily true!

$corr(x,y) = cor(x,y) = 0$

x & y are independent

② ↛ ①

Example: Let X and Y be two discrete random variables with the following join PDF:

		X 0	1	2	
	0	0	0.20	0.10	$f(Y=0)$ =
Y	1	0.20	0.40	0	$f(Y=1)$ =
	2	0.10	0	0	$f(Y=2)$ =
		$f(X=0)$ =	$f(X=1)$ =	$f(X=2)$ =	

What is the correlation between X and Y? And, are X and Y independent?

D-I-Y @ home

1.3.6 Variances of Correlated Variables

Let X and Y be two random variables, then

CORRELATION COEFFICIENT

$$\boxed{\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}}$$

cov(X, Y)

S.D of X

S.D of Y

$$\begin{aligned} var(X+Y) &= var(X) + var(Y) + 2cov(X,Y) \\ &= var(X) + var(Y) + 2\rho\sigma_X\sigma_Y \\ var(X-Y) &= var(X) + var(Y) - 2cov(X,Y) \\ &= var(X) + var(Y) - 2\rho\sigma_X\sigma_Y \end{aligned} \quad (1.6)$$

The generalized result:

Let $\sum_{i=1}^{n} X_i = X_1 + X_2 + \cdots + X_n$, then the variance of the linear combination $\sum X_i$ is:

$$\begin{aligned} var\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} var(X_i) + 2\sum_{i<j}\sum cov(X_i, X_j) \\ &= \sum_{i=1}^{n} var(X_i) + 2\sum_{i<j}\sum \rho_{ij}\sigma_i\sigma_j \end{aligned} \quad (1.7)$$

Example:

what is the $var(X_1 + X_2 + X_3)$?

$$var(x_1 + x_2 + x_3) = var(x_1) + var(x_2) + var(x_3)$$

$$+ 2cov(x_1, x_2) + 2cov(x_2, x_3) + 2cov(x_1, x_3)$$

$$= var(x_1) + var(x_2) + var(x_3)$$

$$+ 2\rho_{12}\sigma_1\sigma_2 + 2\rho_{23}\sigma_2\sigma_3 + 2\rho_{13}\sigma_1\sigma_3 .$$ #

1.3.7 Higher Moments of Probability Distributions

1^{st} moment of PDF

2^{nd} moment of PDF:

$E(X-\mu)^2$

In the previous subsection, we have already discussed about mean, variance, and covariance as the measures of the first and second moments of univariate and multivariate PDFs. Besides the first two moments, we are occasionally interested in the higher moments such as the third and fourth moments which are normally applied in studying the "Shape" of the distribution. In general, the r^{th} moments about the mean is defined as

$$\boxed{r^{th}\textbf{moment} : E(X-\mu)}$$

By the definition of r^{th} moments, we can easily define the third and fourth moments as:

Third moment:

$$E(X-\mu)$$

→ skewness (degree of symmetry)

Fourth moment:

$$E(X-\mu)$$

→ kurtosis (tall / flat)

We can study the shape of the distribution by calculating **skewness** and **kurtosis**.

SKEWNESS is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean.

One measure of skewness is defined as:

$$\boxed{S = \frac{E(X-\mu)^3}{\sigma^3}} = \frac{\text{third moment about the mean}}{\text{cube of the standard deviation}} \tag{1.8}$$

KURTOSIS is a measure of the peakedness of the probability distribution of a real-valued random variable

We can also measure kurtosis as:

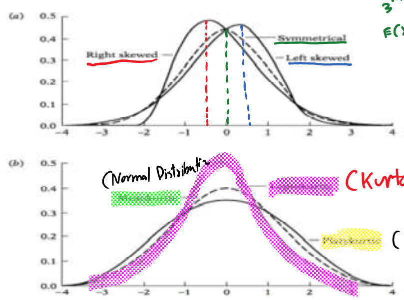
$$
\begin{aligned}
S &= \frac{E(X-\mu)^4}{\sigma^4} \\
&= \frac{\text{fourth moment about the mean}}{\text{square of the second moment}} \qquad (1.9)
\end{aligned}
$$

♣ Platykurtic (fat or short-tailed) ==> PDFs with Kurtosis < 3

♣ **Leptokurtic (slim or long-tailed)** ==> PDFs with Kurtosis > 3

♣ Mesokurtic (which is the normal distribution) ==> PDFs with Kurtosis = 3

3rd moment: $E(x-\mu)^3$



(Kurtosis > 3) tall shape

(Kurtosis < 3) Flat shape

Figure 1.1: (a) Skewness; (b)Kurtosis

Normal ✓

1.4 Some important probability distribution

Chi-square
t-distribution
F-distribution

1.4.1 Normal Distribution

A continuous random variable X has a normal distribution with mean μ and variance σ^2 if its probability density function (pdf) is

exponential

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) \quad \text{where} \quad -\infty < x < \infty$$

$X \sim N(\mu, \sigma^2)$

mean (Centrality)

variance (Dispersion)

NOTE: The normal distribution can be described by two parameters

- μ = The mean of the distribution.
- σ = The standard deviation of the distribution.

Therefore, changing the values of μ and σ alter the positions and shapes of the distributions.

If X is Normally distributed with mean μ and standard deviation σ, we can write it as:

$$\boxed{X \sim N(\mu, \sigma^2)}$$

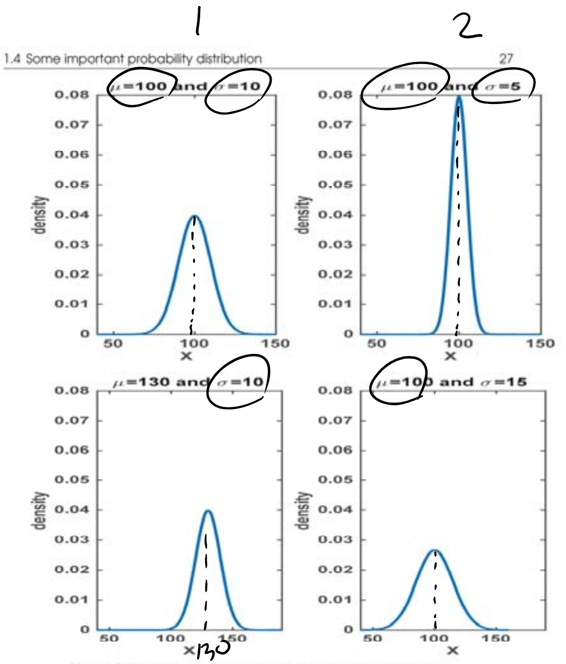


Figure 1.2: Compare the mean and standard deviation of the normal distribution

3 4

The properties of the normal distribution.

★ It is symmetrical around its mean value.

★ About 68 percent of the area under the normal distribution lies between the value $\mu \pm \sigma$

About 95 percent of the area under the normal distribution lies between the value $\mu \pm 2\sigma$

About 99.7 percent of the area under the normal distribution lies between the value $\mu \pm 3\sigma$ (as shown in figure 2)

★ We can convert the given normally distributed variable X with mean μ and σ^2 into the standardized normal variable Z by calculating Z where Z can be defined as:

$$Z = \boxed{\frac{X-\mu}{\sigma}}$$

→ $Z \sim N(0, 1)$
mean S.D.

With the standardized normal variable Z, we can rewrite the normal pdf as:

$$f(Z) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} Z^2\right)$$

In sum, you can see that we convert the given normally distributed variable X into the standardized normal variable by:
(i) Subtracting the mean μ
(ii) Dividing by the standard deviation σ

♡ Subtracting the mean re-centers the distribution on zero.

♡ Dividing by the standard deviation re-scales the distribution so it has standard deviation 1.

It should be remarked that it mean value is zero and its variance is unity for any standardized variable.

By convention, we can denote a normally distributed variable X with zero mean and unit variance as

$$X \sim N(0, 1)$$

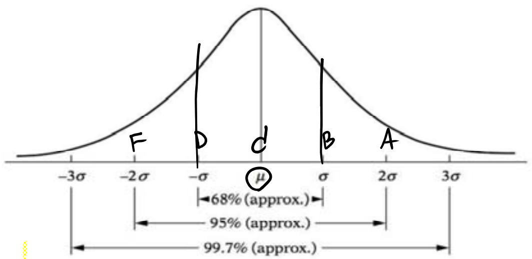


Figure 1.3: Areas under the normal distribution

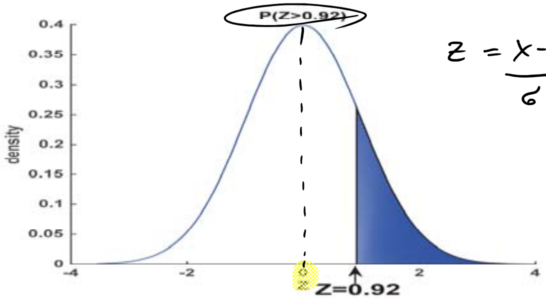
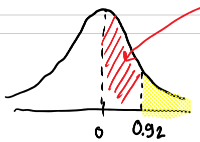
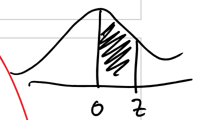


Figure 1.4: If Z ~ N(0,1), the probability that P(Z>0.92)

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

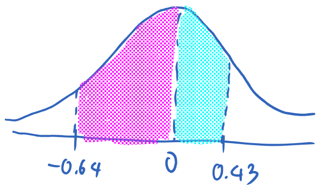




$$P(Z > 0.92) = 0.5 - P[0 < Z \le 0.92]$$
$$= 0.5 - 0.3212$$
$$= 0.1788. \quad \#$$

EX: If $Z \sim N(0,1)$, compute $P(-0.64 \le Z \le 0.43)$.

EX: If $Z \sim N(0,1)$, compute $P(-0.64 \leq Z \leq 0.43)$.



$= P(-0.64 < z < 0) + P(0 \leq z \leq 0.43)$

$= 0.2389 + 0.1664$

$= 0.4053$. #

EX: If $X \sim N(3500, 500^2)$ what is $P(X < 3100)$?

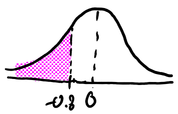
μ ; σ^2

$P(X < 3100)$

$= P\left(\frac{X-\mu}{\sigma} < \frac{3100-3500}{500}\right)$

$= P\left(Z < -\frac{400}{500}\right)$

$= P(Z < -0.8)$



$= 0.5 - P[0 \; Z \leq 0.8]$

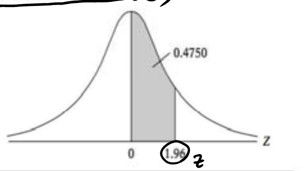
$= 0.5 - 0.2881 = 0.2119$. #

AREAS UNDER THE STANDARDIZED NORMAL DISTRIBUTION (Z)

Example

$\Pr(0 \leq Z \leq 1.96) = 0.4750$

$\Pr(Z \geq 1.96) = 0.5 - 0.4750 = 0.025$



Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4454	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890

1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4454	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ and assume that X_1 and X_2 are independent. If we have the linear combination between X_1 and X_2 where we can write it as:

$$Y = aX_1 + bX_2$$

where a and b are the constant terms. Then

$$Y \sim N\left[(a\mu_1 + b\mu_2), (a^2\sigma_1^2 + b^2\sigma_2^2)\right]$$

In other words, a linear combination of normally distributed variables is itself normally distributed.

Central limit theorem Let $X_1, X_2, \ldots, X_n$ denote n independent random variables and

$$X_i \sim N(\mu, \sigma^2)$$

Let $\bar{X} = \sum \frac{X_i}{n}$, then as n increases indefinitely (i.e. $n \to \infty$),

$$E(\bar{X}) = \frac{E(X_1) + E(X_2) + \ldots + E(X_n)}{n}$$

$$= \frac{n \cdot \mu}{n}$$

$$= \mu. \ \#$$

$$\mathrm{var}(\bar{X}) = \mathrm{var}\left(\frac{X_1 + X_2 + X_3 + \ldots + X_n}{n}\right)$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}. \ \#$$

So, $\boxed{\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)}$. #   variance of $\bar{X}$

if we transform it to Z

Standardized normal

$$Z := \bar{X} - \mu = \sqrt{n}\,(\bar{X} - \mu) \sim N(0, 1). \ \#$$

if we Transform it to Z

Standardized normal variable $Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1)$. #

$\frac{\sigma}{\sqrt{n}}$ S.D. of $\bar{X}$

The third and fourth moments of the normal distribution:

Third moment: $E(X-\mu)^3 = 0$

Fourth moment: $E(X-\mu)^4 = 3\sigma^4$

1.4.2 The χ^2 (Chi-Square) Distribution

Let $Z_1, Z_2, \ldots, Z_k$ be **independent standardized normal variables**. Then the quantity

$$Z = \sum_{i=1}^{k} Z_i^2$$

is said to possess the χ^2 with k degree of freedom (df)

Properties of the χ^2 distribution are as follows:

1.The χ^2 distribution is a skewed distribution where the degree of the skewness depending on the df. As the number of df increases, the distribution becomes more symmetrical. For the df excess of 100, the variable

$$\sqrt{2\chi^2} - \sqrt{(2k-1)}$$

can be converted to a standardized normal variable, where k is the df.

2. The mean of the chi-square distribution is k, and its variance is 2k, where k is the df.

3. If Z_1 and $Z_2$are two independent chi-square variables with k_1 and k_2 df, then the sum of $Z_1 + Z_2$ is also a chi-square with $df = k_1 + k2$

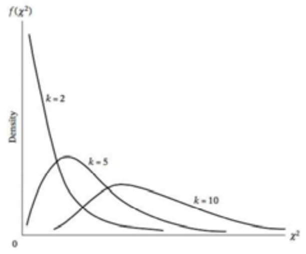


Figure 1.5: Density function of the χ^2 variable

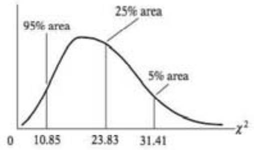
UPPER PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION

Example

$\Pr(\chi^2 > 10.85) = 0.95$

$\Pr(\chi^2 > 23.83) = 0.25$ for df = 20

$\Pr(\chi^2 > 31.41) = 0.05$



Degrees of freedom \ Pr	.995	.990	.975	.950	.900
1	392704×10^{-10}	157088×10^{-9}	982069×10^{-9}	393214×10^{-8}	.0157908
2	.0100251	.0201007	.0506356	.102587	.210720
3	.0717212	.114832	.215795	.351846	.584375
4	.206990	.297110	.484419	.710721	1.063623
5	.411740	.554300	.831211	1.145476	1.61031
6	.675727	.872085	1.237347	1.63539	2.20413
7	.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100*	67.3276	70.0648	74.2219	77.9295	82.3581

*For df greater than 100 the expression $\sqrt{2\chi^2} - \sqrt{(2k-1)} = Z$ follows the standardized normal distribution, where k represents the degrees of freedom.

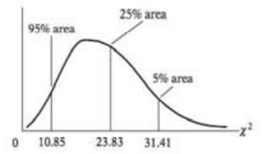
UPPER PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION

Example

$\Pr(\chi^2 > 10.85) = 0.95$

$\Pr(\chi^2 > 23.83) = 0.25$ for df = 20

$\Pr(\chi^2 > 31.41) = 0.05$



.750	.500	.250	.100	.050	.025	.010	.005
.1015308	.454937	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
.575364	1.38629	2.77259	4.60517	5.99147	7.37776	9.21034	10.5966
1.212534	2.36597	4.10835	6.25139	7.81473	9.34840	11.3449	12.8381
1.92255	3.35670	5.38527	7.77944	9.48773	11.1433	13.2767	14.8602
2.67460	4.35146	6.62568	9.23635	11.0705	12.8325	15.0863	16.7496
3.45460	5.34812	7.84080	10.6446	12.5916	14.4494	16.8119	18.5476
4.25485	6.34581	9.03715	12.0170	14.0671	16.0128	18.4753	20.2777
5.07064	7.34412	10.2188	13.3616	15.5073	17.5346	20.0902	21.9550
5.89883	8.34283	11.3887	14.6837	16.9190	19.0228	21.6660	23.5893
6.73720	9.34182	12.5489	15.9871	18.3070	20.4831	23.2093	25.1882
7.58412	10.3410	13.7007	17.2750	19.6751	21.9200	24.7250	26.7569
8.43842	11.3403	14.8454	18.5494	21.0261	23.3367	26.2170	28.2995
9.29906	12.3398	15.9839	19.8119	22.3621	24.7356	27.6883	29.8194
10.1653	13.3393	17.1170	21.0642	23.6848	26.1190	29.1413	31.3193
11.0365	14.3389	18.2451	22.3072	24.9958	27.4884	30.5779	32.8013
11.9122	15.3385	19.3688	23.5418	26.2962	28.8454	31.9999	34.2672
12.7919	16.3381	20.4887	24.7690	27.5871	30.1910	33.4087	35.7185
13.6753	17.3379	21.6049	25.9894	28.8693	31.5264	34.8053	37.1564
14.5620	18.3376	22.7178	27.2036	30.1435	32.8523	36.1908	38.5822
15.4518	19.3374	23.8277	28.4120	31.4104	34.1696	37.5662	39.9968
16.3444	20.3372	24.9348	29.6151	32.6705	35.4789	38.9321	41.4010
17.2396	21.3370	26.0393	30.8133	33.9244	36.7807	40.2894	42.7956
18.1373	22.3369	27.1413	32.0069	35.1725	38.0757	41.6384	44.1813
19.0372	23.3367	28.2412	33.1963	36.4151	39.3641	42.9798	45.5585
19.9393	24.3366	29.3389	34.3816	37.6525	40.6465	44.3141	46.9278
20.8434	25.3364	30.4345	35.5631	38.8852	41.9232	45.6417	48.2899
21.7494	26.3363	31.5284	36.7412	40.1133	43.1944	46.9630	49.6449
22.6572	27.3363	32.6205	37.9159	41.3372	44.4607	48.2782	50.9933
23.5666	28.3362	33.7109	39.0875	42.5569	45.7222	49.5879	52.3356
24.4776	29.3360	34.7998	40.2560	43.7729	46.9792	50.8922	53.6720
33.6603	39.3354	45.6160	51.8050	55.7585	59.3417	63.6907	66.7659
42.9421	49.3349	56.3336	63.1671	67.5048	71.4202	76.1539	79.4900
52.2938	59.3347	66.9814	74.3970	79.0819	83.2976	88.3794	91.9517
61.6983	69.3344	77.5766	85.5271	90.5312	95.0231	100.425	104.215
71.1445	79.3343	88.1303	96.5782	101.879	106.629	112.329	116.321
80.6247	89.3342	98.6499	107.565	113.145	118.136	124.116	128.299
90.1332	99.3341	109.141	118.498	124.342	129.561	135.807	140.169

Source: Abridged from E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians,* vol. 1, 3d ed., table 8, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika.*

1.4.3 Student's t Distribution

If Z_1 is a standardized normal variable and Z_2 is the chi-square distribution with k degree of freedom and is distributed independently of Z_1, then the Student's t distribution (t_k) with k degree of freedom can be represented as

$$\boxed{t = \frac{Z_1}{\sqrt{(Z_2/k)}}}$$
$$= \frac{Z_1\sqrt{k}}{\sqrt{Z_2}} \tag{1.10}$$

Properties of the Student's t distribution are as follows:

1. The t distribution is symmetrical, BUT it is flatter than the normal distribution. However, as the df increase, the t distribution is converted to the normal distribution.

2. The mean of the t distribution is zero, and the variance is $\frac{k}{k-2}$

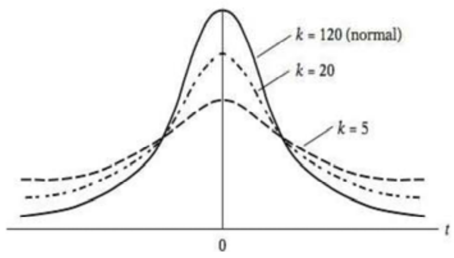


Figure 1.6: Density function of the student's t distribution

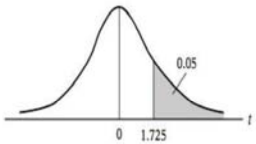
PERCENTAGE POINTS OF THE *t* DISTRIBUTION

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df = 20

$\Pr(|t| > 1.725) = 0.10$



df \ Pr	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
60	0.679	1.296	1.671	2.000	2.390	2.660	3.232
120	0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

1.4.4 The F Distribution

If Z_1 and Z_2 are independently distributed chi-square variables with k_1 and k_2 df, respectively, the (Fisher's) F distribution with k_1 and k_2 df can be written as

$$\boxed{F = \frac{Z_1/k_1}{Z_2/k_2}}$$

The F distribution has the following properties:

1. The F distribution is skewed to the right, but as k_1 and k_2 become large, the F distribution is converted to normal distribution.

2.The mean value of ann F-distributed variable is $\frac{k_2}{(k_2-2)}$, and its variance is

$$\frac{2k_2^2(k_1+k_2-2)}{k_1(k_2-2)^2(k_2-4)}$$

3. The square of a t-distributed random variable with k df is equivalent to an F distribution with 1 and k df.

$$t_k^2 = F_{1,k}$$

4. If the denominator df, k_2, is fairly large, we can get the following relationship

$$k_1 F \sim \chi^2_{k_1}$$

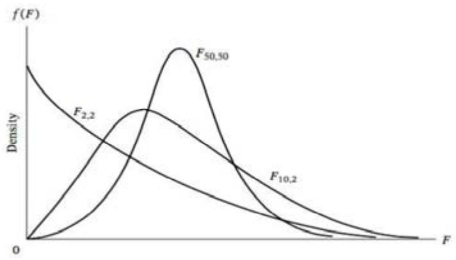


Figure 1.7: Density function of F distribution