

Group 7

Group Homework 5 & 6

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Semester 2/2021 EE320 Introductory mathematical economics

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Due date: April 7th 2022 (before midnight /B.E. moodle).

Q1. Given the production function $Q = f(K, L) = A[K^n + L^n]$ where A is the level of technology, K is capital and L is labor. Suppose that $n > 0$. Consider the following problem.

- a. To ensure that the above production function exhibits a *decreasing return to scale technology*, what additional restrictions do one need to place on n ?

$$Q = f[K, L] = A[K^n + L^n]$$

increase K and L ∂ unit

$$Q^N = A[\partial K^N + \partial L^N]$$

$$= \partial^N [A[K^n + L^n]]$$

$\rightarrow$ Decreasing when $Q^N < Q_{old}$

$N < 1$ because when $N < 1$, it make

∂^N decreases $\rightarrow Q_N < Q_{old}$

- b. Under the assumption used in (a), show that the production function satisfies the law of diminishing returns.

find Marginal Product (MPK)

(K)

$$MPK = \frac{\partial Q}{\partial K}$$

$$Q = AK^n + AL^n$$

$$MPK = \frac{\partial Q}{\partial K} = N \cdot A \cdot K^{N-1}$$

slope of MPK

$$\frac{\partial MPK}{\partial K} = \text{slope}$$

$$\text{slope}_{MPK} = [N-1] \cdot N \cdot A \cdot K^{N-1}$$

from (a) ; $N < 1$

$$\rightarrow [N-1] \cdot N \cdot A \cdot K^{N-1} < 0$$

$$MPK < 0$$



① Marginal Product

$$MPL = \frac{\partial Q}{\partial L} = NAL^{N-1}$$

find slope MPL

$$\text{slope MPL} = \frac{\partial MPL}{\partial L} = N(N-1)AL^{N-2}$$

$$\because N < 1$$

$$\therefore N(N-1)AL^{N-2} < 0$$

Diminishing Return.

- c. Calculate the marginal rate of technical substitution (MRTS) of labor (L) for capital (K).

MRTS = Marginal rate of technical substitution

$$MP_L = \frac{\partial Q}{\partial L}$$

$$\text{given } Q = A[K^N + L^N]$$

$$= AK^N + AL^N$$

$$MP_L = \frac{\partial Q}{\partial L} = NAL^{N-1}$$

$$MP_K = \frac{\partial Q}{\partial K}$$

$$\text{from } Q = A[K^N + L^N]$$

$$= AK^N + AL^N$$

$$MP_K = ANK^{N-1}$$

$$MRTS = \frac{MP_L}{MP_K}$$

$$= \frac{NAL^{N-1}}{ANK^{N-1}} = \left(\frac{L}{K} \right)^{N-1} *$$

- d. Show that MRTS is a decreasing function in L. That is, as labor increases, the value of MRTS decreases.

$$MRTS = \left(\frac{L}{K}\right)^{N-1} = \frac{L^{N-1}}{K^{N-1}}$$

$$\frac{\partial MRTS}{\partial L} = (N-1) \frac{L^{N-2}}{K^{N-1}}$$

↳ from a, $N < 1 \rightarrow N-1$ is negative

$\frac{\partial MRTS}{\partial L}$ is negative : L increase, MRTS decrease
L decrease, MRTS increase

- e. Under the condition(s) assumed in (a), is the production function satisfied with the *global* concave property?

(K)

$$Q = AK^n + AL^n$$

$$\frac{\partial Q}{\partial K} = nAK^{n-1}$$

$$\frac{\partial^2 Q}{\partial K^2} = n(n-1)AK^{n-2}$$

↳ from a, $n < 1 \rightarrow$ SOC is negative
 $\therefore$ concave

(L)

$$Q = AK^n - AL^N$$

$$\frac{\partial Q}{\partial L} = nAL^{n-1}$$

↙ negative

$$\frac{\partial^2 Q}{\partial L^2} = n(n-1)AL^{n-2}$$

SOC is negative
 $\therefore$ concave

Suppose that $K(t) = \frac{1}{2}t^2 + 2t + 3$ and $L(t) = e^t + 3$, where $t \geq 0$ is the number of periods from now. Consider the following problem

f. Show that Q is increasing over time.

$$\begin{aligned} Q &= f(K, L) ; \quad \frac{dQ}{dt} = \frac{\partial Q}{\partial K} K'(t) + \frac{\partial Q}{\partial L} L'(t) \\ &= A n K^{n-1} \cdot \frac{dK}{dt} + A n L^{n-1} \cdot \frac{dL}{dt} \\ &= A n K^{n-1} (t+2) + A n L^{n-1} (te^{t-1}) \end{aligned}$$

$\therefore Q$ is increasing over time $*$

g. Compute $\frac{\bar{Q}}{dt}$ when $t = 0$, i.e. growth of output in the initial period.

$$\begin{aligned} \left. \frac{dQ}{dt} \right|_{t=0} & \quad t = 0 ; \\ & K(t) = \frac{1}{2}(0^2) + 2(0) + 3 = 3 \\ & L(t) = e^0 + 3 = 4 \end{aligned}$$

$$\begin{aligned} \left. \frac{dQ}{dt} \right|_{t=0} &= A n K^{n-1} (t+2) + A n L^{n-1} (te^{t-1}) \\ &= A n K^{n-1} (0+2) + A n L^{n-1} (0e^{0-1}) \\ &= A n (3)^{n-1} \cdot 2 + A n 4^{n-1} \cdot (0) \\ &= 2 A n (3)^{n-1} \quad * \end{aligned}$$

Q2. A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

a. What type of the return to scale technology does the production function exhibit?

From now on, assume that $c = \frac{1}{4}$. Consider the following problems.

Cobb - Douglas Production Function : $Q = K^{\alpha} \cdot L^{\beta}$

If $\alpha + \beta < 1$, there will be decreasing return to scale.

$\alpha + \beta = 1$, $\rightarrow$ constant return to scale.

$\alpha + \beta > 1$, $\rightarrow$ increasing return to scale.

$$\therefore Q = K^{\frac{1}{3}}L^{\frac{2}{3}}, \quad \alpha + \beta = \frac{1}{3} + \frac{2}{3} = 1$$

$\therefore$ The production function is constant return to scale. #

b. Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$P = Q^{-c}, \quad Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$$

TC_K : Total Cost of Capital

$$\text{Market Demand : } P = Q^{-c}$$

TC_L : Total Cost of Labor

$$\text{Substitute } c = \frac{1}{4} \rightarrow P = Q^{-\frac{1}{4}}$$

$$\text{Profit} = TR - TC$$

$$\pi = P(Q) \cdot Q - (TC_K + TC_L)$$

$$\pi = Q^{-\frac{1}{4}} \cdot Q - [Kr + wL]$$

The unit price of K = r

K unit price = Kr

The unit price of w = w

L unit price = wL

$$\pi = Q^{-\frac{1}{4}} \cdot Q^1 - [Kr + wL]$$

$$\pi = Q^{-\frac{1}{4}} \cdot Q^{\frac{4}{4}} - Kr - wL$$

$$\pi = Q^{\frac{3}{4}} - Kr - wL$$

$$\text{From } Q = K^{\frac{1}{3}} \cdot L^{\frac{2}{3}}$$

$$\pi = [K^{\frac{1}{3}} \cdot L^{\frac{2}{3}}]^{\frac{3}{4}} - Kr - wL$$

$$\pi = (K^{\frac{1}{3}})^{\frac{3}{4}} \cdot (L^{\frac{2}{3}})^{\frac{3}{4}} - Kr - wL$$

$$\pi = K^{\frac{1}{4}} \cdot L^{\frac{1}{2}} - Kr - wL$$

$$f(K, L) = K^{\frac{1}{4}} \cdot L^{\frac{1}{2}} - Kr - wL \quad \#$$

c. The firm wants to maximize profit and seek for combination of the two factor inputs. Derive the demand for factor inputs, capital and labor.

FOC:

$$\frac{df}{dK} = 0, \quad \frac{df}{dL} = 0$$

$$\frac{df}{dK} = \frac{1}{4} K^{\frac{1}{4}-\frac{4}{4}} \cdot L^{\frac{1}{2}} - (1)r$$

$$\frac{df}{dK} = \frac{1}{4} K^{-\frac{3}{4}} L^{\frac{1}{2}} - r$$

$$\frac{df}{dK} = \frac{L^{\frac{1}{2}}}{4K^{\frac{3}{4}}} - r$$

$$0 = \frac{L^{\frac{1}{2}}}{4K^{\frac{3}{4}}} - r$$

$$r = \frac{L^{\frac{1}{2}}}{4K^{\frac{3}{4}}}$$

$$4r = \frac{L^{\frac{1}{2}}}{K^{\frac{3}{4}}} \quad \text{--- ①}$$

$$\frac{df}{dL} = K^{\frac{1}{4}} \cdot \left[\frac{1}{2} L^{\frac{1}{2}-\frac{2}{2}} \right] - 0 - w(1)$$

$$\frac{df}{dL} = K^{\frac{1}{4}} \cdot \frac{1}{2} L^{-\frac{1}{2}} - w$$

$$\frac{df}{dL} = K^{\frac{1}{4}} \left(\frac{1}{2L^{\frac{1}{2}}} \right) - w$$

$$0 = \frac{K^{\frac{1}{4}}}{2L^{\frac{1}{2}}} - w$$

$$2w = \frac{K^{\frac{1}{4}}}{L^{\frac{1}{2}}} \quad \text{--- ②}$$

$$\text{①} \times \text{②} ; \left(\frac{L^{\frac{1}{2}}}{K^{\frac{3}{4}}} \right) \left(\frac{K^{\frac{1}{4}}}{L^{\frac{1}{2}}} \right) = (4r)(2w)$$

$$\frac{K^{\frac{1}{4}}}{K^{\frac{3}{4}}} = 8rw$$

$$K^{-\frac{2}{4}} = 8rw$$

$$K^{-\frac{1}{2}} = 8rw$$

$$\frac{1}{8rw} = K^{\frac{1}{2}}$$

$$\left(K^{\frac{1}{2}} \right)^2 = \left(\frac{1}{8rw} \right)^2$$

$$K = \frac{1}{64r^2w^2} \quad \text{--- ③}$$

Substitute K in ①

$$L^{\frac{1}{2}} / K^{\frac{3}{4}} = 4r$$

$$L^{\frac{1}{2}} = K^{\frac{3}{4}} (4r)$$

$$L^{\frac{1}{2}} = \left[\frac{1}{64r^2w^2} \right]^{\frac{3}{4}} (4r)$$

$$L^{\frac{1}{2}} = \frac{1}{(64)^{\frac{3}{4}} (r^2)^{\frac{3}{4}} (w^2)^{\frac{3}{4}}} \cdot 4r$$

$$L^{\frac{1}{2}} = \frac{4r}{(2^6)^{\frac{3}{4}} \cdot r^{\frac{3}{2}} \cdot w^{\frac{3}{2}}}$$

$$L^{\frac{1}{2}} = \frac{4r}{2^{\frac{9}{2}} \cdot r^{\frac{3}{2}} \cdot w^{\frac{3}{2}}}$$

$$(L^{\frac{1}{2}})^2 = \left(\frac{4r}{2^{\frac{9}{2}} \cdot r^{\frac{3}{2}} \cdot w^{\frac{3}{2}}} \right)^2$$

$$L = \frac{16r^2}{2^9 \cdot r^3 \cdot w^3}$$

$$L = \frac{\cancel{2^4} \cancel{r^2}}{\cancel{2^4} \cdot \cancel{r^2} \cdot w^3} = \frac{1}{2^5 r w^3}$$

$$L = \frac{1}{32rw^3} \text{ --- ④}$$

Production function : $Q = K^{\frac{1}{3}} L^{\frac{2}{3}}$

$$Q = \left(\frac{1}{64r^2w^2} \right)^{\frac{1}{3}} \left(\frac{1}{32rw^3} \right)^{\frac{2}{3}}$$

$$Q = \frac{1}{(64)^{\frac{1}{3}} (r^2)^{\frac{1}{3}} (w^2)^{\frac{1}{3}} \cdot (32)^{\frac{2}{3}} (r)^{\frac{2}{3}} \cdot w^2}$$

$$Q = \frac{1}{\sqrt[3]{32} \sqrt[3]{2} r^{\frac{4}{3}} w^{\frac{8}{3}}} \quad \#$$

d. How does the demand for labor vary with respect to w and r ? Show your result by using partial derivative.

$$L = \frac{1}{32rw^3}$$

$$= \frac{1}{32} \cdot r^{-1} \cdot w^{-3}$$

$$\frac{dL}{dr} = \frac{1}{32} \cdot (-1r^{-2}) \cdot w^{-3}$$

$$\frac{dL}{dr} = \frac{-1}{32r^2w^3} \quad \#$$

$$\frac{dL}{dw} = \frac{1}{32} r^{-1} [-3w^{-4}]$$

$$\frac{dL}{dw} = \frac{-3}{32rw^4} \quad \#$$

e. Confirm your answer with the second-order condition.

soc. Find f_{KK} , f_{KL} , f_{LK} , f_{LL}

$$f_K = \frac{1}{4} K^{-\frac{3}{4}} L^{\frac{1}{2}} - r$$

$$f_{KK} = \frac{1}{4} \left(\frac{3}{4} K^{-\frac{3}{4}-\frac{4}{4}} \right) L^{\frac{1}{2}} - 0$$

$$f_{KK} = -\frac{3}{16} K^{-\frac{7}{4}} L^{\frac{1}{2}}$$

$$f_{KK} = \frac{-3L^{\frac{1}{2}}}{16K^{\frac{7}{4}}} \quad \#$$

$$f_{KL} = \frac{1}{4} K^{-\frac{3}{4}} \left(\frac{1}{2} L^{-\frac{1}{2}} \right) - 0$$

$$f_{KL} = \frac{1}{8K^{\frac{3}{4}} L^{\frac{1}{2}}} \quad \#$$

$$f_L = \frac{1}{2} K^{\frac{1}{4}} L^{-\frac{1}{2}} - w$$

$$f_{LK} = \frac{1}{2} \left(\frac{1}{4} K^{\frac{1}{4}-\frac{4}{4}} \right) L^{-\frac{1}{2}} - 0$$

$$f_{LK} = \frac{K^{-\frac{3}{4}} L^{-\frac{1}{2}}}{8}$$

$$f_{LK} = \frac{1}{8K^{\frac{3}{4}} L^{\frac{1}{2}}} \quad \#$$

$$f_{LL} = \frac{1}{2} K^{\frac{1}{4}} \left(-\frac{1}{2} L^{-\frac{1}{2}-\frac{2}{2}} \right) - 0$$

$$f_{LL} = \frac{-K^{\frac{1}{4}} L^{-\frac{3}{2}}}{4}$$

$$f_{LL} = -\frac{K^{\frac{1}{4}}}{4L^{\frac{3}{2}}} \quad \#$$

$$H = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{-3L^{\frac{1}{2}}}{16K^{\frac{7}{4}}} & \frac{1}{8r^{\frac{3}{4}}L^{\frac{1}{2}}} \\ \frac{1}{8K^{\frac{3}{4}}L^{\frac{1}{2}}} & \frac{-K^{\frac{1}{4}}}{4L^{\frac{3}{2}}} \end{bmatrix}$$

Substitute $K = \frac{1}{64r^2w^2}$, $L = \frac{1}{32rw^3}$

$$L^{\frac{1}{2}} = \left(\frac{1}{32rw^3} \right)^{\frac{1}{2}}$$

$$L^{\frac{1}{2}} = \frac{1}{2^{\frac{5}{2}} r^{\frac{1}{2}} w^{\frac{3}{2}}} \quad \#$$

$$K^{\frac{7}{4}} = \left(\frac{1}{64r^2w^2} \right)^{\frac{7}{4}}$$

$$= \frac{1}{2^{\frac{21}{2}} \cdot r^{\frac{7}{2}} \cdot w^{\frac{7}{2}}} \quad \#$$

$$K^{\frac{3}{4}} = \left(\frac{1}{64r^2w^2} \right)^{\frac{3}{4}}$$

$$= \frac{1}{2^{\frac{9}{2}} \cdot r^{\frac{3}{2}} \cdot w^{\frac{3}{2}}} \quad \#$$

a. What type of the return to scale technology does the production function exhibit?

From now on, assume that $c = 1/4$. Consider the following problems.

b. Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L .)

c. The firm wants to maximize profit and seek for combination of the two factor inputs. Derive the demand for factor inputs, capital and labor.

d. How does the demand for labor vary with respect to w and r ? Show your result by using partial derivative.

e. Confirm your answer with the second-order condition.

Q3. In a duopoly market, suppose that firms 1 and 2 face market demand, $p = 100 - (q_1 + q_2)$. Firm costs are $c_1 = 10q_1$ and $c_2 = q_2^2$.

(a) Calculate market price under the Cournot environment. How much is the profit that each firm yields in the equilibrium?

(b) Calculate the market price under the joint production decision, i.e. collusive equilibrium. How much is the profit that each firm yields in the equilibrium?

(c) Do you think that the collusive equilibrium will sustain? What would be the profit of firm 1 if firm 1 chooses to deviate from the collusive allocation?

Q4. Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Q3. In a duopoly market, suppose that firms 1 and 2 face market demand, $p = 100 - (q_1 + q_2)$. Firm costs are $c_1 = 10q_1$ and $c_2 = q_2^2$.

(a) Calculate market price under the Cournot environment. How much is the profit that each firm yields in the equilibrium?

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(c) Do you think that the collusive equilibrium will sustain? What would be the profit of firm 1 if firm 1 chooses to deviate from the collusive allocation?

① Cournot environment

$$\begin{aligned}\text{Profit}_1 &= P \cdot q_1 - C_1 \\ &= [100 - (q_1 + q_2)] q_1 - 10 q_1 \\ &= 100 q_1 - q_1^2 - q_1 q_2 - 10 q_1\end{aligned}$$

$$\pi_1 = 90 q_1 - q_1^2 - q_1 q_2$$

$$\begin{aligned}\max \pi_1 \rightarrow \frac{d\pi}{dq_1} &= 90 - 2q_1 - q_2 = 0 \\ 90 - q_2 &= 2q_1 \\ q_1 &= 45 - 0.5q_2 \quad \textcircled{1}\end{aligned}$$

$$\begin{aligned}\text{Profit}_2 &= P \cdot q_2 - C_2 \\ &= [100 - (q_1 + q_2)] q_2 - q_2^2 \\ &= 100 q_2 - q_1 q_2 - q_2^2 - q_2^2\end{aligned}$$

$$\pi_2 = 100 q_2 - q_1 q_2 - 2q_2^2$$

$$\max \pi_2 : \frac{d\pi}{dq_2} = 100 - q_1 - 4q_2 = 0$$

$$q_1 = 100 - 4q_2 \quad \textcircled{2}$$

② into ①

$$100 - 4q_2 = 45 - 0.5q_2$$

$$100 - 45 = -0.5q_2 + 4q_2$$

$$55 = 3.5q_2$$

$$q_2 = 15.71$$

$$q_1 = 100 - 4(15.71)$$

$$q_1 = 37.16$$

$$Q_{\text{sum}} = 37.16 + 15.71$$

$$= 52.87$$

$$P_{\text{price}} = 100 - 52.87 = 47.13$$

$$\pi_1 = 90q_1 - q_1^2 - q_1q_2$$

$$= 90(37.16) - 15.71^2 - 37.16(15.71)$$

$$= 3344.4 - 246.8 - 583.8$$

$$= 2513.8$$

$$\pi_2 = 100q_2 - q_1q_2 - 2q_2^2$$

$$= 100(15.71) - 37.16(15.71) - 2(15.71)^2$$

$$= 1571 - 583.78 - 493.6$$

$$= 493.62$$

b

$$\pi_{\text{sum}} = \pi_1 + \pi_2$$

$$= P \cdot Q - C_1 - C_2$$

$$= (100 - q_1 - q_2)(q_1 + q_2) - 10q_1 - q_2^2$$

$$= 100q_1 - q_1^2 - q_1q_2 + 100q_2 - q_1q_2 - q_2^2 - 10q_1 - q_2^2$$

$$\pi^T = 90q_1 - q_1^2 - 2q_1q_2 + 100q_2 - 2q_2^2$$

$$\frac{\partial \pi}{\partial q_1} = 90 - 2q_2 - 2q_1 = 0$$

$$q_1 = 45 - q_2$$

$$\frac{\partial \pi}{\partial q_2} = -2q_1 + 100 - 4q_2 = 0$$

$$2q_1 = 100 - 4q_2$$

$$q_1 = 50 - 2q_2$$

plg ① in ②

$$45 - q_2 = 50 - 2q_2$$

$$2q_2 - q_2 = 50 - 45$$

$$q_2 = 5$$

$$q_1 = 40$$

$$P = 100 - 45 = 55$$

$$\pi_1 = P \cdot q_1 - C_1 = 55(40) - 10(40) = 1,800$$

$$\pi_2 = P \cdot q_2 - C_2 = 55(5) - 5^2 = 250$$

③

$$\begin{array}{l} \pi_1 = 2513.8 \\ \pi_2 = 492.98 \end{array} \quad \left. \vphantom{\begin{array}{l} \pi_1 \\ \pi_2 \end{array}} \right\} \text{Cournot}$$

$$\begin{array}{l} \pi_1 = 1800 \\ \pi_2 = 250 \end{array} \quad \left. \vphantom{\begin{array}{l} \pi_1 \\ \pi_2 \end{array}} \right\} \text{collusion}$$

$$j \quad \pi_1 \text{ collusion} < \pi_1 \text{ Cournot}$$

firm₁ might not collusion (diviate)

$$\pi_2 \text{ collusion} < \pi_1 \text{ Cournot}$$

firm₂ also not collusion (diviate)

$\therefore$ collusion not stable

Q4. Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market equilibrium

Market A : $D = S$

$$10 - 2Q_A = 1 + Q_A$$

$$Q_A = 3$$

$$P_A = 4$$

Market B : $D = S$

$$20 - Q_B = 2 + 2Q_B$$

$$Q_B = 6$$

$$P_B = 14$$

หรือ ราคาที่ผู้บริโภคจ่าย
 (b) $t = P^D - P^S$
 ราคาที่ผู้ผลิตรับได้

Market A

$$t_A = [10 - 2Q_A] - [1 + Q_A]$$

$$t_A = 9 - 3Q_A$$

$$Q_A = \frac{9 - t_A}{3} \quad P_A = 1 + \frac{9 - t_A}{3}$$

Market B

$$t_B = [20 - Q_B] - [2 + 2Q_B]$$

$$t_B = 18 - 3Q_B$$

$$Q_B = \frac{18 - t_B}{3} \quad P_B = 2 + 2\left(\frac{18 - t_B}{3}\right)$$

(c) tax revenue = $t_A + t_B$

$$\text{Tax R} = t_A \cdot Q_A + t_B \cdot Q_B$$

$$\text{Tax R} = t_A \left[\frac{9 - t_A}{3} \right] + t_B \left[\frac{18 - t_B}{3} \right]$$

(d) Max

$$\text{Tax} = 3t_A - \frac{t_A^2}{3} + 6t_B - \frac{t_B^2}{3}$$

$$\frac{\partial \text{Tax}}{\partial t_A} ; 3 - \frac{2}{3}t_A = 0$$

$$t_A = 4.5 \quad \times$$

$$\frac{\partial \text{Tax}}{\partial t_B} ; 6 - \frac{2}{3}t_B = 0$$

$$t_B = 9 \quad \times$$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

- Derive the market equilibrium
- Suppose the government imposes unit tax on both markets at the rate of t_A and t_B . Solve for the after-tax equilibrium as the function of t_A and t_B .
- How much revenue can the government collect from the taxation?
- Determine the level of t_A and t_B that maximizes government's revenue.

Q5. (Monopolist and the Optimal advertising)

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. Both θ and β are strictly positive.

The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

- Construct the profit function.
- Determine the optimal pricing (P^*) and optimal advertisement (A^*) that maximize the profit of the monopolist.
- Confirm the result with the second-order differential test, i.e. hessian-matrix method.

$$a) \pi = P(Q) \cdot Q - C(Q) - A$$

$$\begin{aligned} \text{Sub in } Q &= 2000 + 4\sqrt{A} - 20P; &= P(2000 + 4\sqrt{A} - 20P) - 2Q - 1000 - A \\ & &= 2000P + 4P\sqrt{A} - 20P^2 - 4000 - 8\sqrt{A} + 40P \\ & &\quad - 1000 - A \end{aligned}$$

$$= 2040P + 4PA^{\frac{1}{2}} - 20P^2 - 5000 - 8A^{\frac{1}{2}} - A$$

b) Determine the optimal pricing (P^*) and optimal advertisement (A^*) that maximize the profit of the monopolist.

$$\begin{aligned} P: 2040 + 4A^{\frac{1}{2}} - 40P &= 0 \\ 40P - 4A^{\frac{1}{2}} &= 2040 \\ P^* &= \frac{2040 + 4A^{\frac{1}{2}}}{40} \# \end{aligned}$$

$$A: 2PA^{-\frac{1}{2}} - 4A^{-\frac{1}{2}} - 1 = 0$$

$$\frac{2P}{A^{\frac{1}{2}}} - \frac{4}{A^{\frac{1}{2}}} = 1$$

$$2P - 4 = A^{\frac{1}{2}}$$

$$4P^2 - 16 = A^* \#$$

Sub in A^*

$$P^* = \frac{2040 + 4(4P^2 - 16)^{\frac{1}{2}}}{40}$$

$$= \frac{2040 + 4(2P - 4)}{40}$$

$$= \frac{2040 + 8P - 16}{40}$$

$$40P = 2040 + 8P - 16$$

$$32P = 2024$$

$$P^* = 63.25 \#$$

sub into A^* ;

$$A^* = 4(63.25)^2 - 16$$

$$= 15986.25 \#$$

c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.

$$H = \begin{bmatrix} \pi_{PP} & \pi_{PA} \\ \pi_{AP} & \pi_{AA} \end{bmatrix} = \begin{bmatrix} -40 & 2A^{-\frac{1}{2}} \\ 2A^{-\frac{1}{2}} & A^{-\frac{3}{2}}(2-P) \end{bmatrix}$$

$$|H_1| = -40 < 0 \rightarrow \forall (P, A)$$

$$|H_2| = -40(A^{-\frac{3}{2}}(2-P)) - (2A^{-\frac{1}{2}})^2 < 0 \rightarrow \forall (P, A)$$

H is negative definite $\forall (P, A)$. π is globally concave

P^*, A^* are our global solution.

The profit maximization bundle #