Heteroscedasticity

When σ_i^2 is not known

White's Heteroscedasticity-Consistent Variances and Standard Errors

- White has shown that this estimate can be performed so that asymptotically valid statistical inferences can be made about the true parameter values

- White's heteroscedascity-corrected variances and standard errors are known as robust standard errors

Plausible Assumptions about Heteroscedasticity Pattern

To illustrates this, let us revert to the two variable regression model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (1)$$

We now consider several assumptions about the pattern of heteroscedasticity.

When σ_i^2 is not known

Several assumptions about the pattern of heteroscedasticity

Assumption 1 The error variance is proportional to X_i^2

$$E(u_i^2) = \sigma^2 X_i^2$$

It is believed that the variance of u_i is proportional to the square of the explanatory variable X, one may transform the original model as follows.

$$\frac{Y_i}{X_i} = \frac{\beta_1}{X_i} + \beta_2 + \frac{u_i}{X_i} \qquad (2)$$

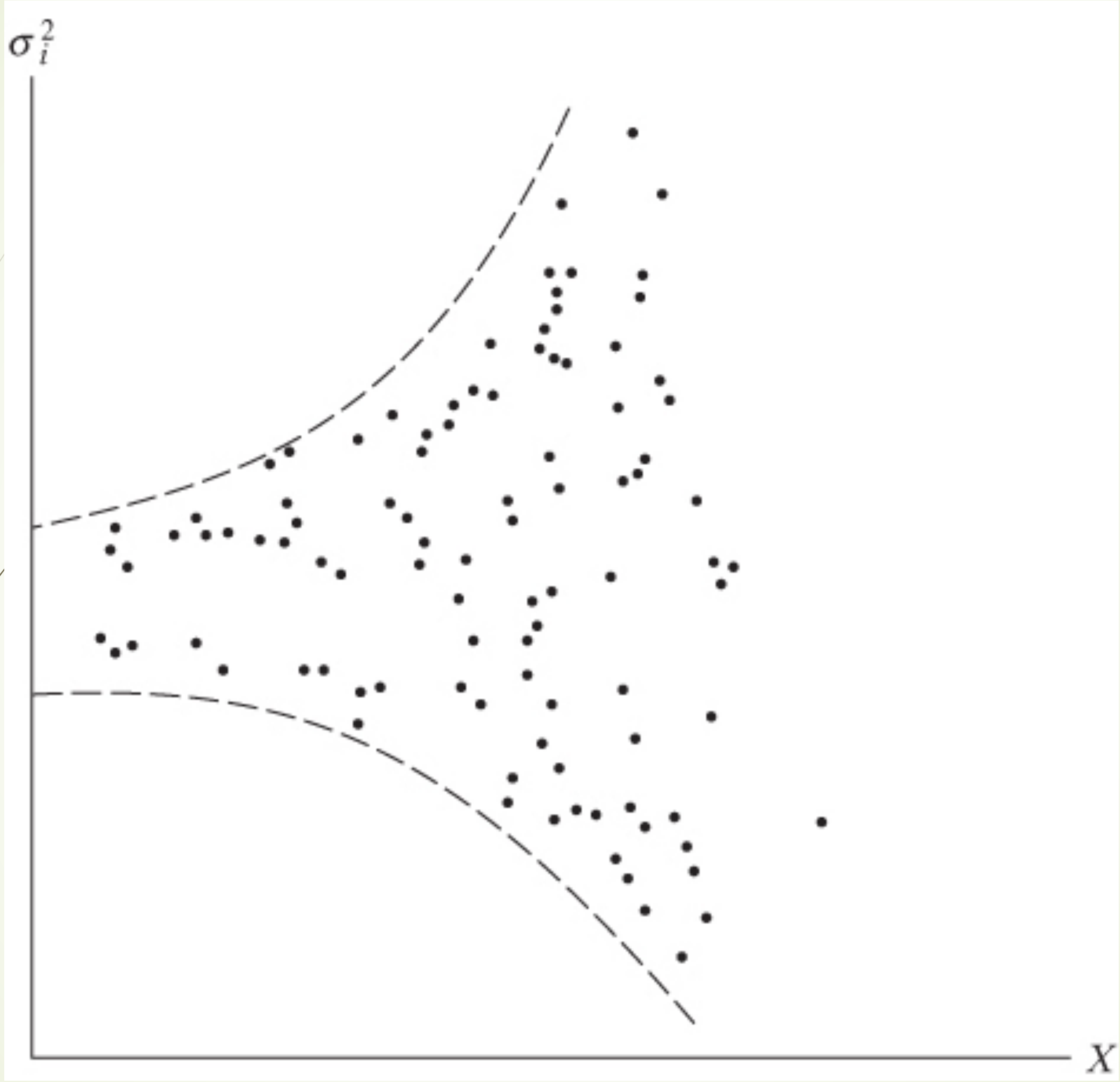
$$= \beta_1 \frac{1}{X_i} + \beta_2 + \nu_i$$

Where v_i is the transformed disturbance term, equal to $\frac{u_i}{X_i}$. Now it is easy to verify that

$$E(\nu_i^2) = E\left(\frac{u_i}{X_i}\right)^2 = \frac{1}{X_i^2} E(u_i^2)$$

$$= \sigma^2$$

Hence, the variance of v_i is now homoscedastic, and one may proceed to apply OLS to the transformed equation (2)


σ_i^2
X

Assumption 2 The error variance is proportional to X_i

The **square root transformation:**

$$E(u_i^2) = \sigma^2 X_i$$

It is believed that the variance of u_i is proportional to the explanatory variable X, one may transform the original model as follows.

$$\frac{Y_i}{\sqrt{X_i}} = \frac{\beta_1}{\sqrt{X_i}} + \beta_2\sqrt{X_i} + \frac{u_i}{\sqrt{X_i}} \qquad (3)$$

$$= \beta_1 \frac{1}{\sqrt{X_i}} + \beta_2\sqrt{X_i} + v_i$$

Where $v_i = \frac{u_i}{\sqrt{X_i}}$ and where $X_i > 0$

Given assumption 2, one can verify that $E\left({v_i}^2\right)$=σ^2, a homoscedasticity situation

Therefore, one may proceed to apply OLS to equation (3)

- Note an important feature of the transformed model: it has no intercept term.

- Therefore, one will have to use the regression through the origin model to estimate β_1 and β_2.

- Having run (3), one can get back to the original model simply by multiplying (3) by $\sqrt{X_i}$

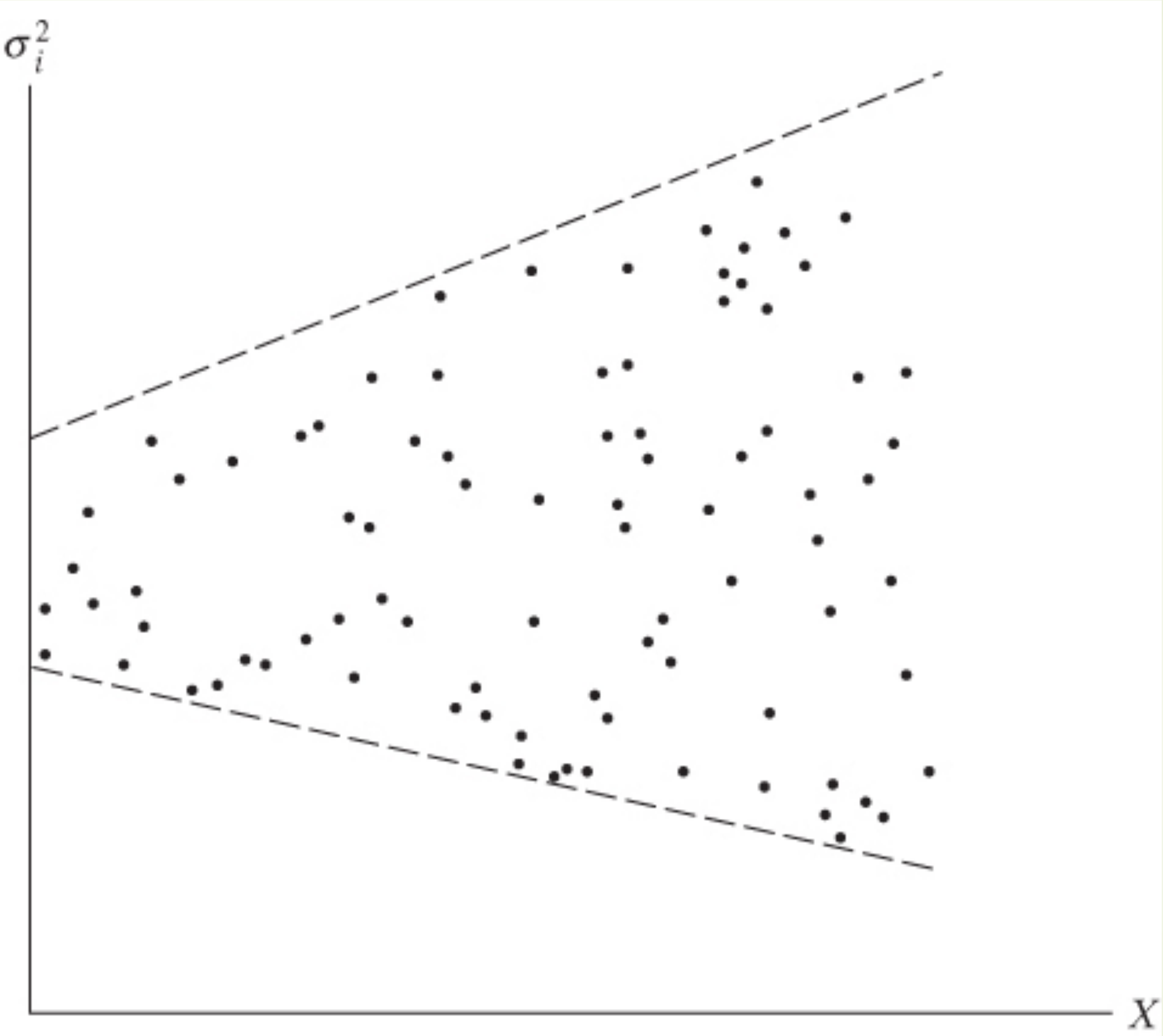
- An interesting case is the zero intercept model, $Y_i = \beta_2 X_i + u_i$. In this case, equation (3) becomes:

$$Y_i = \beta_2 \sqrt{X_i} + \frac{u_i}{\sqrt{X_i}}$$

- It can be shown that

$$\hat{\beta}_2 = \frac{\bar{Y}}{\bar{X}}$$

That is the weighted least squares estimator is simply the ratio of the means of the dependent and explanatory variables


σ_i^2
X

Assumption 3 The error variance is proportional to the square of the mean value of Y

$$E(u_i^2) = \sigma^2 \left[E(Y_i) \right]^2$$

This postulates that variance of u_i is proportional to the square of the expected value of Y. Now

$$E(Y_i) = \beta_1 + \beta_2 X_i$$

- Therefore, it we transform the original equation as follows,

$$\frac{Y_i}{E(Y_i)} = \frac{\beta_1}{E(Y_i)} + \beta_2 \frac{X_i}{E(Y_i)} + \frac{u_i}{E(Y_i)}$$

$$= \beta_1 \left(\frac{1}{E(Y_i)} \right) + \beta_2 \frac{X_i}{E(Y_i)} + \nu_i \qquad (4)$$

Where $v_i = \frac{u_i}{E(Y_i)}$, it can be seen that $E(v_i) = \sigma^2$

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$$

$$\frac{Y_i}{\hat{Y}_i} = \beta_1 \left(\frac{1}{\hat{Y}_i} \right) + \beta_2 \left(\frac{X_i}{\hat{Y}_i} \right) + \nu_i \qquad (5)$$

Where $v_i = \frac{u_i}{\widehat{Y_i}}$

The transformation (4) is, however, inoperational because $E(Y_i)$ depends on β_1 and β_2, which are unknown.

We know $\widehat{Y_i} = \hat{\beta}_1 + \hat{\beta}_2 X_i$, which is an estimator of $E(Y_i)$. Therefore, we may proceed in two steps:

- Run the OLS regression, disregarding the heteroscedasticity problem and obtain $\widehat{Y_i}$
- Using the estimated $\widehat{Y_i}$, we transform our model as (5)

Although $\widehat{Y_i}$ are not exactly $E(Y_i)$, they are consistent estimators; that is as the sample size increases indefinitely, they converge to the true $E(Y_i)$. Hence equation (5) will perform satisfactory in practice if the sample size is large.

Assumption 4 A log transformation such as

$$lnY_i = \beta_1 + \beta_2 lnX_i + u_i$$

Very often reduces heteroscedasticity when compared with the regression $Y_i = \beta_1 + \beta_2 X_i + u_i$

- The result arises because log transformation compresses the scales in which the variables are measured

Additional problems with the transformations we have considered that should be borne in mind:

- When we go beyond the two-variable model, we may not know a priori which of the X variables should be chosen for transforming the data
- Log transformation is not applicable if some of the Y and X values are zero or negative
- The problem of spurious correlation
- When σ_i are not directly known and are estimated from one or more of the transformations that we have discussed earlier, all our testing procedures using the t tests, F tests, etc. are strictly speaking, valid only in large samples. Therefore, one has to be careful in interpreting the results based on the various transformation in small or finite samples.