

Chapter 5

Nonlinear Model and Differential Calculus in Economic Theory

Topics:

- Slope and derivatives of a function
- Rule of differentiation
- Non differentiable functions
- Convexity and concavity
- Inflection point
- Maxima-Minima
- Examples in Economics
 - Relations among the total, the average and the marginal functions
 - Derivative and marginality
 - Elasticity



Comparative Statics and Derivative

VOCAB: “Difference Quotient” / “Derivative”/ “Differentiation”/ “Differential Calculus”

Comparative Statics is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables.

For example,

When $G = G_0$

- (1.) $Y = C + I + G$
- (2.) $C = a + bY_d$
- (3.) $Y_d = Y - T$
- (4.) $I = I_0$
- (5.) $G = G_0$

$$Y_E|_{G=G_0} = \frac{a - bT + I_0 + G_0}{1 - b}$$

$$Y_E = f(\text{exo}, \text{parameters})$$

$$\frac{\partial Y_E}{\partial \text{exo}}, \frac{\partial Y_E}{\partial \text{parameters}}$$

When $G = G_1$

- (1.) $Y = C + I + G$
- (2.) $C = a + bY_d$
- (3.) $Y_d = Y - T$
- (4.) $I = I_0$
- (5.) $G = G_1$

$$Y_E|_{G=G_1} = \frac{a - bT + I_0 + G_1}{1 - b}$$

$$\rightarrow \frac{\partial Y_E}{\partial a}, \frac{\partial Y_E}{\partial b}, \frac{\partial Y_E}{\partial T_0}, \frac{\partial Y_E}{\partial I_0}, \frac{\partial Y_E}{\partial G_1}$$

when $G_0 \rightarrow Y_E|_{G_0}$

when $G_1 \rightarrow Y_E|_{G_1}$

$$\Delta G = G_1 - G_0 \Rightarrow \Delta Y = Y_E|_{G_1} - Y_E|_{G_0}$$

It should be clear that the problem under consideration is essentially one of finding a rate of change: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogenous variable.

The notion of rate of change is directly concerned with the mathematical concept of *derivative*, in *differential calculus*.

Looking at a function: $y = f(x)$

$$\text{when } x_1 \rightarrow y_1 = f(x_1)$$

$$\text{when } x_2 \rightarrow y_2 = f(x_2)$$

$$\Delta x = x_2 - x_1 \Rightarrow \Delta y = y_2 - y_1$$

We are interested in $\frac{\Delta y}{\Delta x}$ which is called “difference quotient”

When $\Delta x \rightarrow 0$, $\frac{\Delta y}{\Delta x} \rightarrow \frac{dy}{dx}$.

That is, $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = f'(x)$

$\frac{dy}{dx}$ or $f'(x)$ is called the derivative of f at x .

Notation for derivative of f at x : $f'(x)$, $\frac{df(x)}{dx}$, $\frac{dy}{dx}$, $D_x f(x)$, y' .

A *derivative* is a function. The word *derivative* means a *derived function*, from the original function $y = f(x)$, which is also called a *primitive function*.

When we have a Primitive function $y = f(x)$ and try to find the derived function $\frac{dy}{dx}$, we call this process “differentiation”.

Derivative and differential Calculus will be used in finding maximum, minimum points and in optimization problem.



The Slope of a Curve and The Derivative of a function

The concept of the **slope** of a curve is the geometric counterpart of the concept of the **derivative**. Both concepts deal with the **marginal** notion used in economics.

The slope of a total cost function: $C = F(q)$ measures the change in total cost resulting from a unit increase in output, i.e. the marginal cost(MC). That is,

$$MC(Q) = \frac{\Delta C}{\Delta q}$$

And when $\Delta q \rightarrow 0$

$$MC(Q) = \frac{dC}{dq}$$

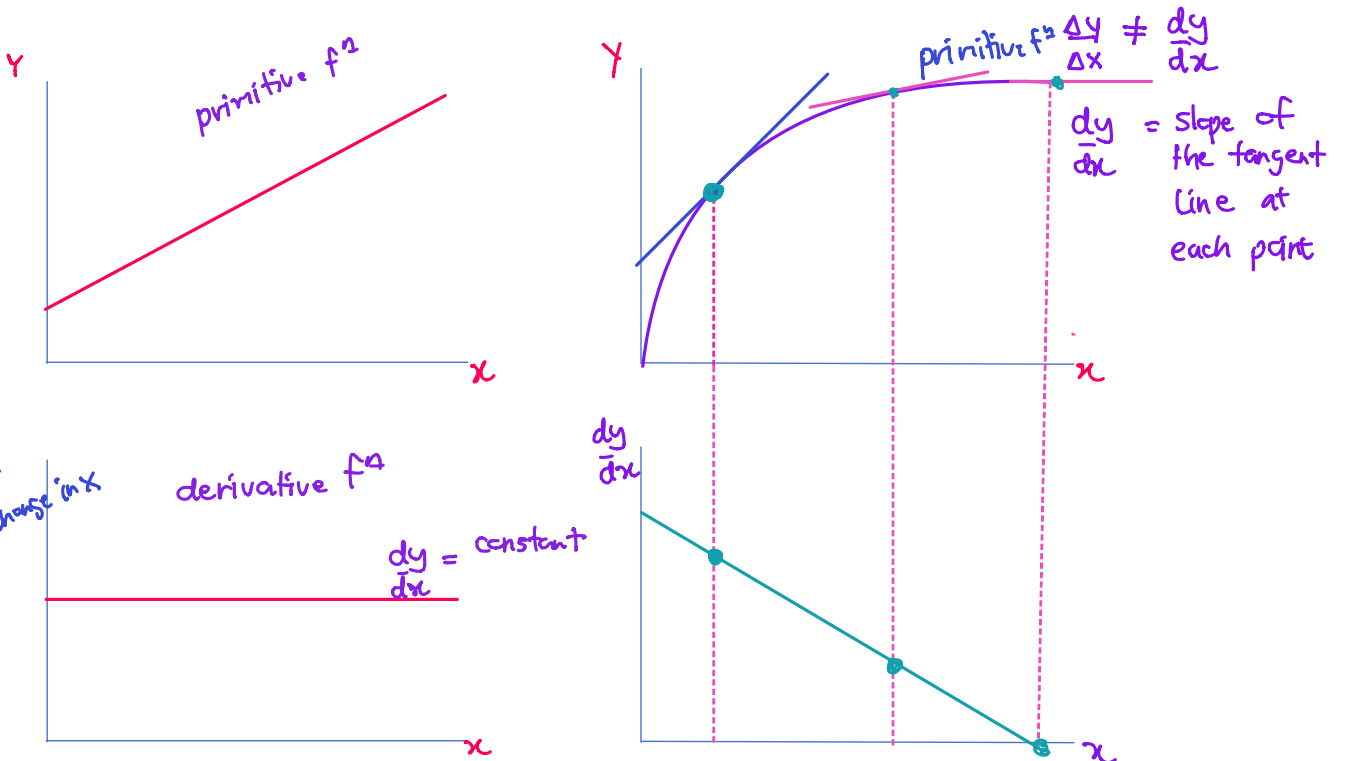
The slope of a utility function: $u = U(x)$ measures the change in utility resulting from a unit increase in consumption, i.e. the marginal utility(MU). That is,

$$MU(x) = \frac{\Delta u}{\Delta x}$$

And when $\Delta x \rightarrow 0$

$$MU(x) = \frac{du}{dx}$$

- ⊙ If a function is linear, the slope is constant and is equal for every points on the linear curve.
- ⊙ If a function is nonlinear function, the slope is not constant. Slope for each point on the curve might not be equal. Slope at each point on a nonlinear function is the slope of the tangent line at that point.



Differentiation / derivation / process of obtaining the derivative

From definition of derivative, we can find the derivative from a primitive function as the following.

For example, let $f(x) = 2x^2 + 4$ find $f'(x)$:

(1.) $f'(x) = 4x$

(2.) Use derivative definition: $f'(x) = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$

Let x increases from x_1 to $x_2 = x_1 + h$, so $\Delta x = x_2 - x_1 = h$ with

$$f(x_1) = 2x_1^2 + 4 \quad \text{and} \quad f(x_2) = 2x_2^2 + 4 = 2(x_1 + h)^2 + 4 = 2x_1^2 + 4x_1h + 2h^2 + 4$$

Therefore,

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{h} \\ &= \frac{(2x_1^2 + 4x_1h + 2h^2 + 4) - (2x_1^2 + 4)}{h} \\ &= 4x_1 + 2h \end{aligned}$$

x_1 is actually can be any $x \quad \frac{\Delta y}{\Delta x} = 4x + 2h$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right) \\ &= \lim_{h \rightarrow 0} (4x + 2h) \\ &= 4x \quad \# \end{aligned}$$

Differentiability of a function

“A function is differentiable at point x_0 if it is smooth and continuous at point x_0 .”

★ Continuity

🔔 Continuity is a necessary condition for a function to be differentiable.

The function f is continuous at a if

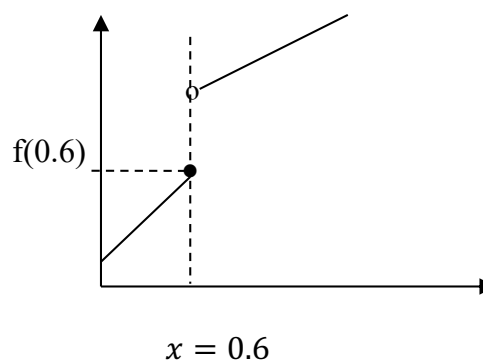
1. We can find $f(a)$, i.e. $x = a$ must be in the domain of the function f .
2. We can find $\lim_{x \rightarrow a} f(x)$
3. $\lim_{x \rightarrow a} f(x) = f(a)$

Example of a function that is not continuous at $x = 0.6$

$f(x)$

เนื่องจาก

1. $f(0.6)$ can be found.
 2. $\lim_{x \rightarrow 0.6^-} f(x) \neq \lim_{x \rightarrow 0.6^+} f(x)$
- $\therefore f(x)$ is not continuous at $x = 0.6$



H.w.: Is this function continuous $f(x) = \begin{cases} x^2 & \text{when } x < 2 \\ x+1 & \text{when } x \geq 2 \end{cases}$?

★ Smooth Function(has no kink)

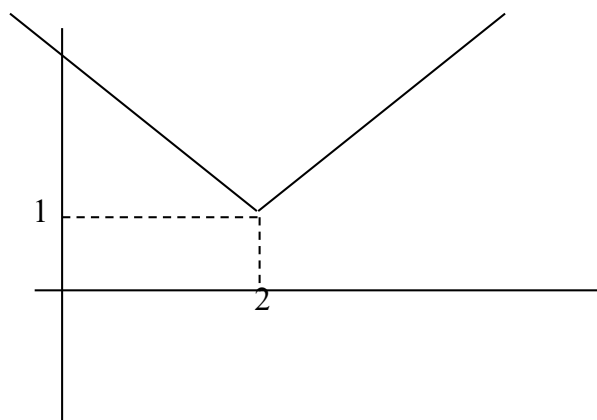
🔔 Smooth function is a sufficient condition for differentiability.

The differentiability condition is:

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Example of a function that has kink: $f(x) = |x - 2| + 1$

$f(x)$



Continuity: $f(2) = \lim_{x \rightarrow 2} f(x) = 1$, $f(x)$ is continuous at $x = 2$

But if we try to find derivative:

Function f has derivative at a if:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If we let $x = a + h$

when $h \rightarrow 0$, $x \rightarrow a$ and we can rewrite $f'(a)$ as:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}, \quad x \neq a$$

Derivative of $f(x)$ at $x = 2$ is:

$$f'(2) = \lim_{x \rightarrow 2} \frac{|x-2|+1-1}{x-2} = \lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$$

$$\text{where } \lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} = \frac{-(x-2)}{x-2} = -1$$

$$\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = \frac{x-2}{x-2} = 1$$

$$\therefore \lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} \neq \lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$$

Therefore, $f'(2) = \lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ cannot be found.

Even $f(x)$ is continuous at $x = 2$ but we cannot find $f'(2)$ because $f(x)$ has a kink at $x = 2$.

Notation:

$f \in c^{(0)}$ or $f \in c$: means f is continuous.

$f \in c^{(1)}$ or $f \in c'$: means f is continuously differentiable (A function f with a continuous derivative function, i.e. the everywhere-smooth function)

Rule of Differentiation

1. If $f(x) = c$, c is a constant $f'(x) = 0$
2. If $f(x) = cg(x)$, c is a constant $f'(x) = cg'(x)$
3. If $f(x) = x^n$, n is any real number $f'(x) = nx^{n-1}$
4. If $f(x) = U(x) \pm V(x)$, $f'(x) = U'(x) \pm V'(x)$
5. If $f(x) = U(x)V(x)$, $f'(x) = U(x)V'(x) + V(x)U'(x)$
6. If $f(x) = \frac{U(x)}{V(x)}$, $f'(x) = \frac{V(x)U'(x) - U(x)V'(x)}{[V(x)]^2}$
7. [chain rule] If we have a differentiable function $z = U(y)$ and another differentiable function $y = V(x)$, then the derivative of z with respect to x is equal to the derivative of z with respect to y , times the derivative of y with respect to x .

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = U'(y)V'(x)$$

Change in x determines change in y via function V , and change in y determines change in z via function U .

8. [Derivatives of Inverse Function] Let $y = f(x)$, we have $\frac{dy}{dx}$. The inverse function of f , $x = f^{-1}(y)$, its derivative is $\frac{dx}{dy}$, and

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

9. [Derivatives of log function]

$$\text{If } y = \log_a x, \frac{dy}{dx} = \frac{1}{x \ln a}.$$

$$\text{If } y = \ln x, \frac{dy}{dx} = \frac{1}{x \ln e} = \frac{1}{x}.$$

$$\text{If } y = \ln V(x), \frac{dy}{dx} = \frac{V'(x)}{V(x)}.$$

10. [Derivatives of exponential function]

$$\text{If } y = a^x, \text{ where } a > 0, a \neq 1, \frac{dy}{dx} = a^x \ln a$$

$$\text{If } y = e^x, \frac{dy}{dx} = e^x$$

$$\text{If } y = e^{V(x)}, \frac{dy}{dx} = e^{V(x)} V'(x)$$

H.w.:

(a.) find $f'(x)$ for the following functions:

$$f(x) = \sqrt{2x^{-2}}$$

$$f(x) = 2x^3 + 3x^2 - 5x + 1$$

$$f(x) = (2x + 3)(3x^2)$$

$$f(x) = \frac{ax^2 + b}{cx}$$

$$f(x) = (x - 4x^2)^3$$

$$f(x) = (1 - x^2)\sqrt{1 - 2x}$$

(b.)

If $y = f(x) = 5x + 25$, find $\frac{dx}{dy}$

If $y = f(x) = x^5 + x$, find $\frac{dx}{dy}$

(c.)

Find $f'(x)$ if $f(x) = e^{3x-1}$

Find $f'(x)$ if $f(x) = \ln(1 + 2x + x^2)$

Find $f'(x)$ if $f(x) = (1 + x)(1 + e^{x^2})(3 - x)^{\frac{1}{2}}$

Find $f'(x)$ if $f(x) = (x^2 - 1)\sqrt{\frac{2 - 3x^2}{1 - 2x^3}}$

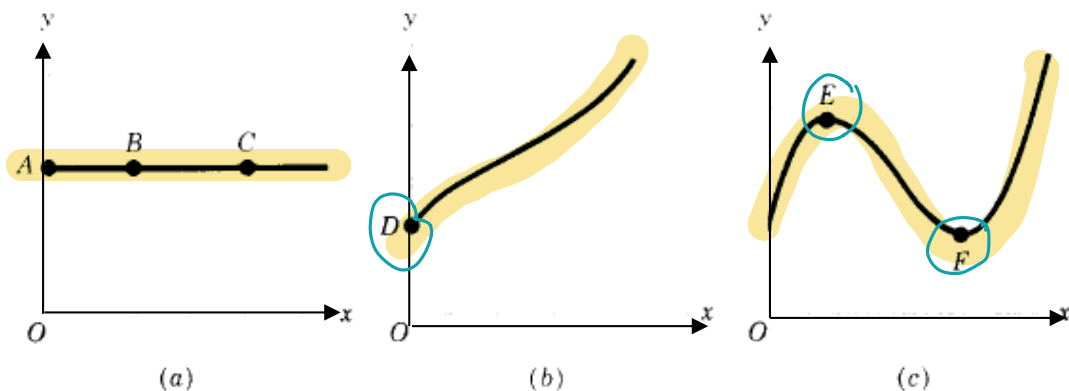
(d.) Let $y = x^4 - x^{\frac{4}{3}} + 6x^{\frac{1}{3}}$, find the second derivative of $f(x)$ with respect to x , $f''(x)$.



Maxima and Minima, Convexity and Concavity

Global vs. Local Extremum Concept

Consider (a), (b), (c) and (d)



Graph(a.): Constant Function

No maximum/minimum of y

$$\frac{dy}{dx} = 0$$

Graph (b.): Strictly Increasing Function

When x increases, y increases. There is no maximum value of y , and the minimum value of y is at point D.

Note:

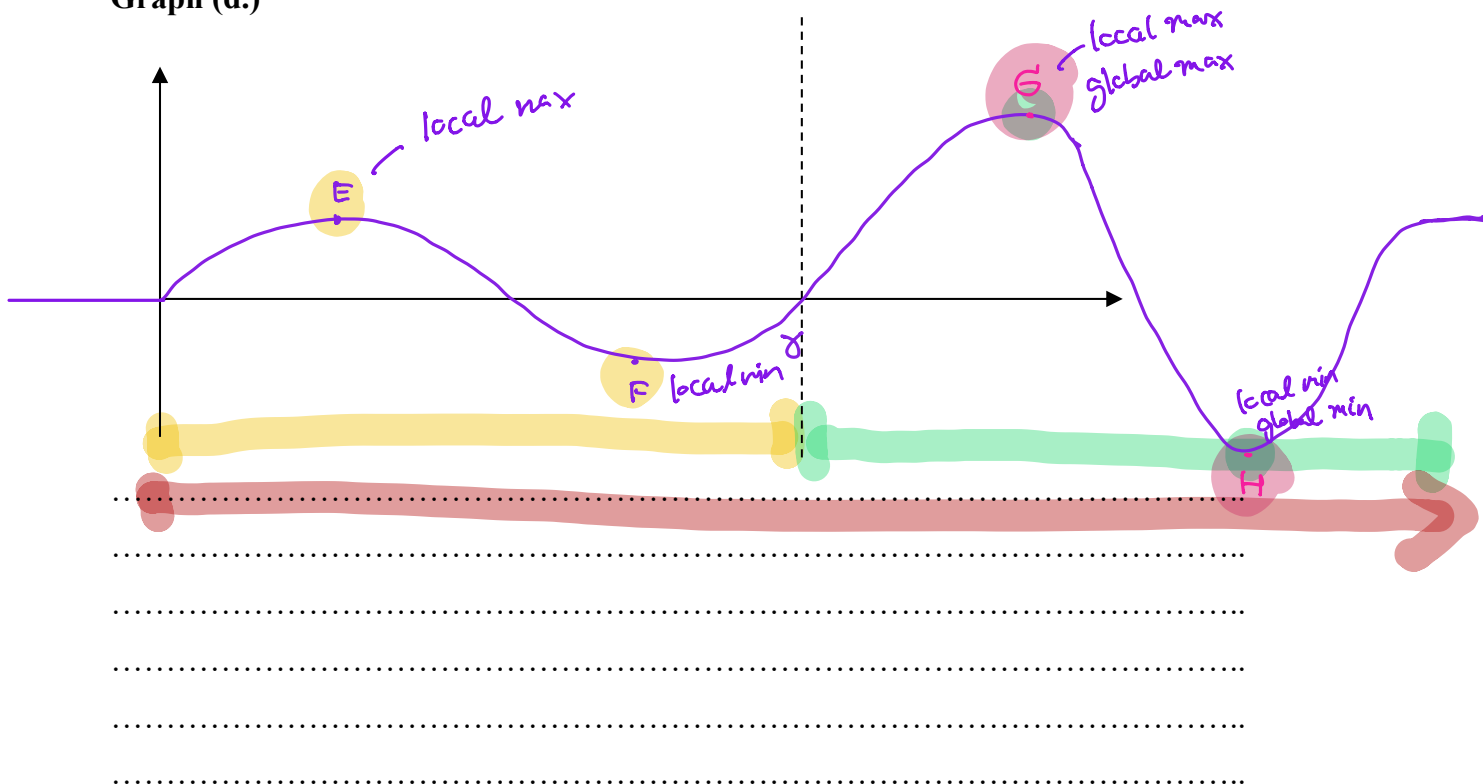
Monotonic increasing function is when $\frac{dy}{dx} \geq 0$.

Strictly monotonic increasing function is when $\frac{dy}{dx} > 0$.

Monotonic decreasing function is when $\frac{dy}{dx} \leq 0$.

Strictly monotonic decreasing function is when $\frac{dy}{dx} < 0$.

Graph (d.)



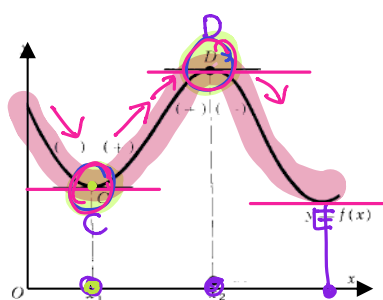
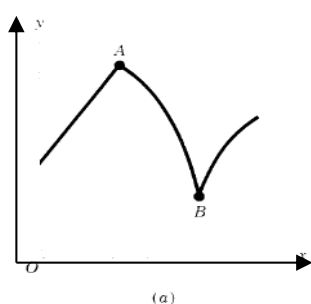
How can we test if a critical point is a local max, a local min, or an inflection point?

★ First-Derivative Test ★

If $y = f(x)$ has Local Max/ Local Min at $x = x_0$, possible cases for the first derivative are:

(a.) $f'(x_0)$ does not exist

(b.) $f'(x_0) = 0$



C, D, E are stationary points

critical point / value

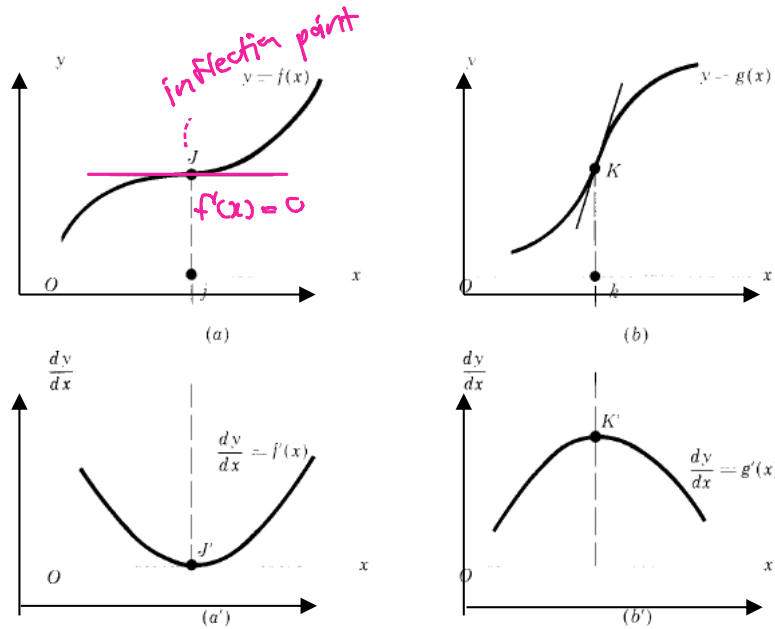
If $f(x)$ is a smooth function, the necessary condition for a point to be a local max or local min is:

$$\frac{df(x)}{dx} = \frac{dy}{dx} = \underline{f'(x) = 0}$$

as x changes in an extremely small unit $f(x)$ doesn't change.

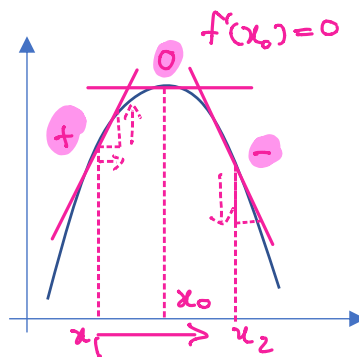
We call the order pair (x, y) that satisfies this necessary condition a stationary point, in which x is “critical point or critical value”, and y is “stationary value”.

The condition $f'(x) = 0$ is a necessary condition, but not a sufficient condition) for a point to be a local maximum or local minimum.



If the first derivative of $f(x)$ at $x = x_0$ is 0 ($f'(x_0) = 0$), then $f(x_0)$ will be

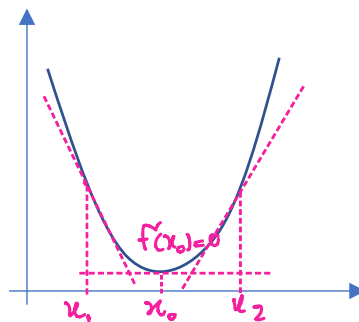
1. **Local maximum**, when the first derivative changes from being positive to negative when $x < x_0$ and $x > x_0$



when $x < x_0$, $f'(x_1) > 0$

when $x > x_0$, $f'(x_2) < 0$

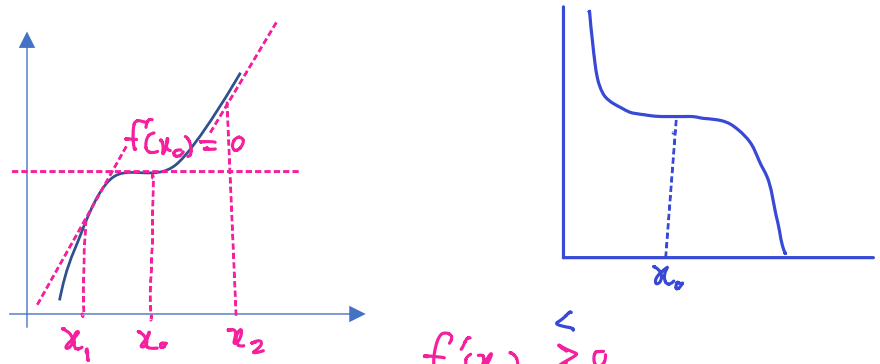
2. **Local minimum**, when the first derivative changes from being negative to positive when $x < x_0$ and $x > x_0$



when $x < x_0$, $f'(x_1) < 0$

when $x > x_0$, $f'(x_2) > 0$

3. **Inflection point**, when the first derivative doesn't change sign when $x < x_0$ and $x > x_0$



when $x < x_0$, $f'(x_1) < 0$

when $x > x_0$, $f'(x_2) > 0$

H.w.: Find local maxima or minima of the following functions:

$$y = f(x) = x^3 - 12x^2 + 36x + 8$$

Average-Cost Function $AC = f(Q) = Q^2 - 5Q + 8$

⊛ **Second-Derivative Test** ⊛

Interpretation of the Second Derivative $f(x)$

$f'(x)$ slope of graph $y = f(x)$

$f''(x)$ rate of change of slope of graph $y = f(x)$
slope of graph $y = f'(x)$

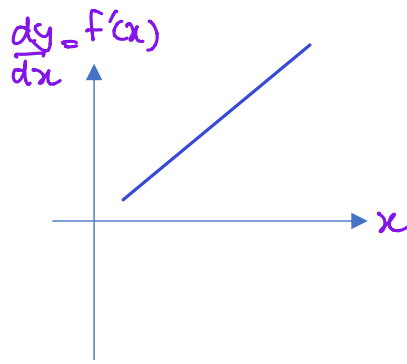
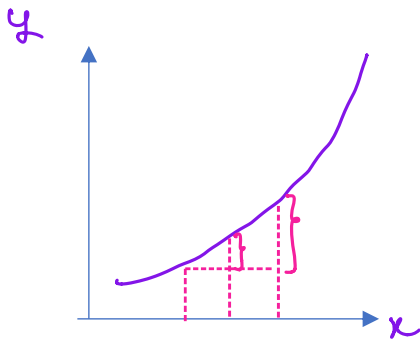
$f'(x) > 0$
 $f'(x) < 0$ } $f(x)$ will be } strictly increasing
strictly decreasing

$f''(x) > 0$
 $f''(x) < 0$ } slope of $f(x)$ will be } strictly increasing
strictly decreasing



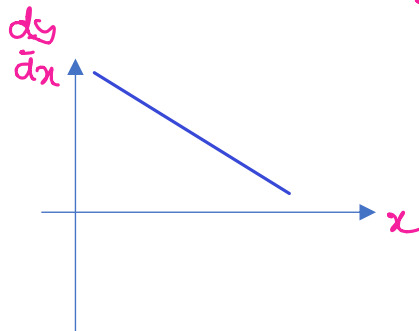
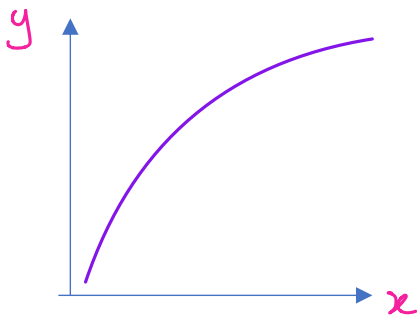
Consider $f(x)$ such that

1) If $f'(x) > 0$ and $f''(x) > 0$,



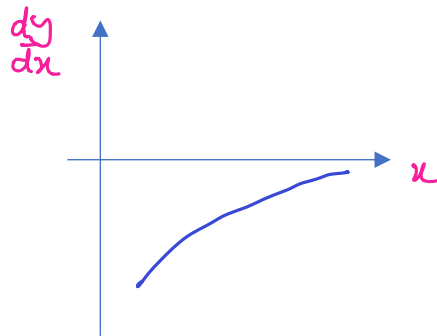
as $x \uparrow$, $y \uparrow$ at increasing rate

2) If $f'(x) > 0$ and $f''(x) < 0$,



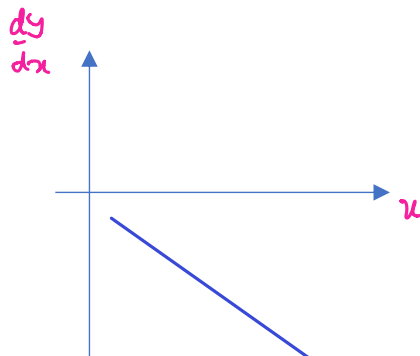
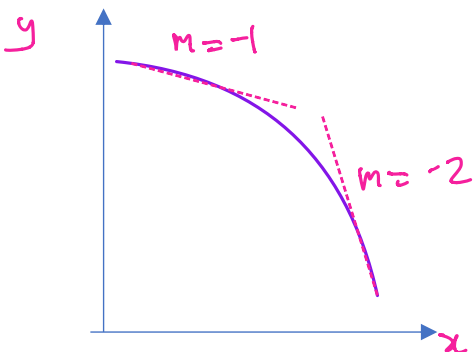
as $x \uparrow$, $y \uparrow$ at decreasing rate

3) If $f'(x) < 0$ and $f''(x) > 0$,



as $x \uparrow$, $y \downarrow$ at decreasing rate (magnitude of slope \downarrow) (the less negative) slope is

4) If $f'(x) < 0$ and $f''(x) < 0$,



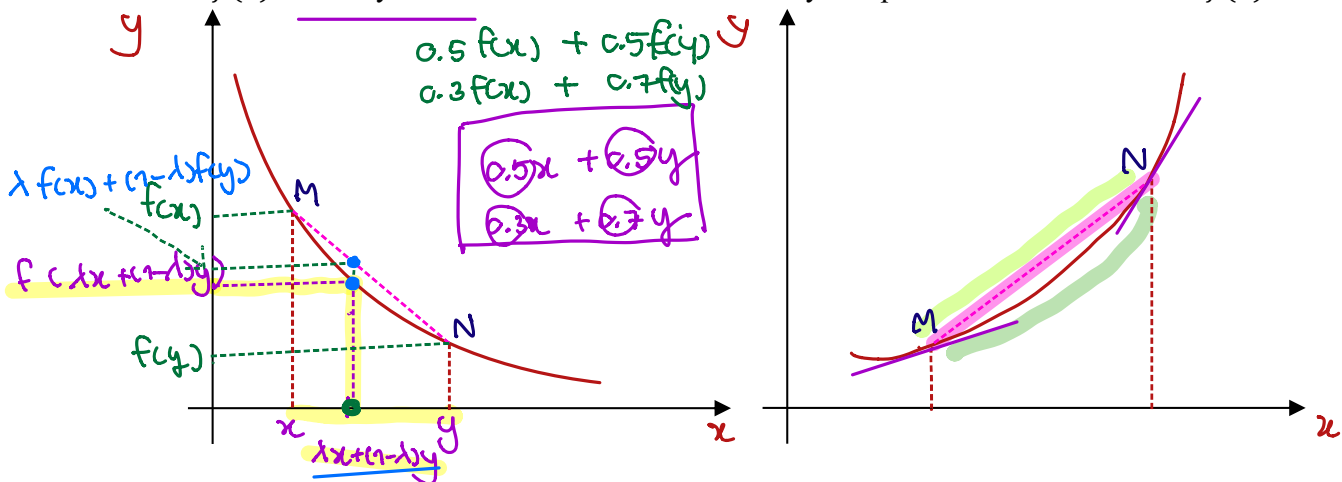
as $x \uparrow$, $y \downarrow$ at increasing rate the slope is more negative (magnitude of slope \uparrow)

Curvature of a Graph

A graph can be:

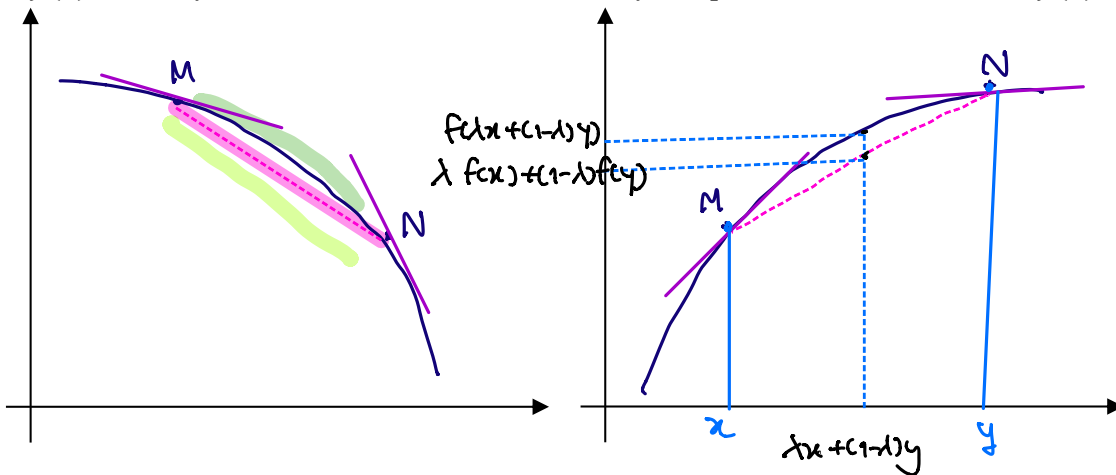
- **Strictly Convex** Convex e.g. risk-loving utility f^u
 $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly convex if $\forall x, y \in \mathbb{R}, \lambda \in (0, 1)$
 $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$
 f^u of lin com $<$ lin com of f^u
 f^u of lin com \leq lin com of f^u
- **Strictly Concave/Concave** e.g. risk-averse utility f^u
 $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly concave if $\forall x, y \in \mathbb{R}, \lambda \in (0, 1)$
 $f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$
 f^u of lin com $>$ lin com of f^u

$f(x)$ is strictly convex if a linear line between any two points M and N lies above $f(x)$

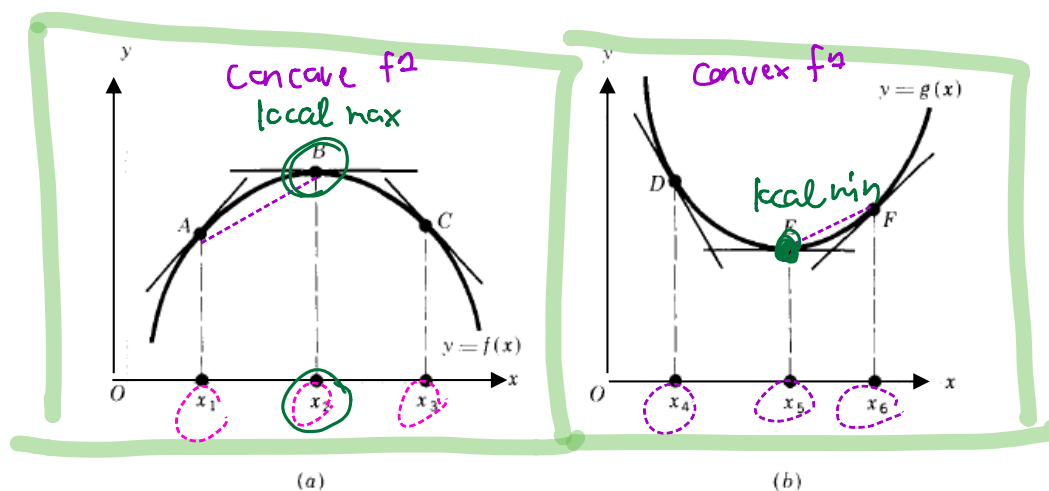


$f''(x) > 0$ for strictly convex function

$f(x)$ is strictly concave if a linear line between any two points M and N lies below $f(x)$



$f''(x) < 0$ for strictly concave function



From graph (a.)

$$x_1 < x_2 < x_3$$

$$f'(x_1) > f'(x_2) > f'(x_3)$$

$$x \uparrow, f'(x) \downarrow$$

$$f''(x) < 0$$

$$f''(x_2) < 0$$

From graph (b.)

$$x_4 < x_5 < x_6$$

$$f'(x_4) < f'(x_5) < f'(x_6)$$

$$x \uparrow, f'(x) \uparrow$$

$$f''(x) > 0$$

$$f''(x_5) > 0$$

If $f'(x_0) = 0$, $f(x_0)$ will be

→ a local maximum when..... $f''(x_0) < 0$

→ a local minimum when..... $f''(x_0) > 0$

for $f(x)$: an objective f²at x_0 a critical value

SUMMARY

Condition	Maximum	Minimum
<u>First-order necessary</u> FONC, FCC	$f'(x_0) = 0$	$f'(x_0) = 0$
<u>Second-order necessary</u> SONC	$f''(x_0) \leq 0$	$f''(x_0) \geq 0$
<u>Second-order sufficient</u> SOSC, SOC	$f''(x_0) < 0$	$f''(x_0) > 0$

H.W.:

(1.) $f(x) = \frac{1}{8}(x^4 - 8x^2)$

(2.) $f(x) = \frac{x^4}{4} - \frac{3}{2}x^2$

(3.) $f(x) = x^4$



Application of differential calculus in economics

✈ Total Cost, Average Cost, and Marginal Cost

$$TC = C(Q)$$

$$AC = \frac{TC(Q)}{Q} \Rightarrow TC(Q) = AC(Q) \times Q$$

$$MC = \frac{dTC(Q)}{dQ}$$

$$= \frac{d(AC(Q) \times Q)}{dQ}$$

$$MC = AC(Q) + Q \frac{dAC(Q)}{dQ}$$

$$MC - AC = Q \frac{dAC}{dQ} \quad \text{---} \quad \text{☆}$$

Relationship between MC and AC

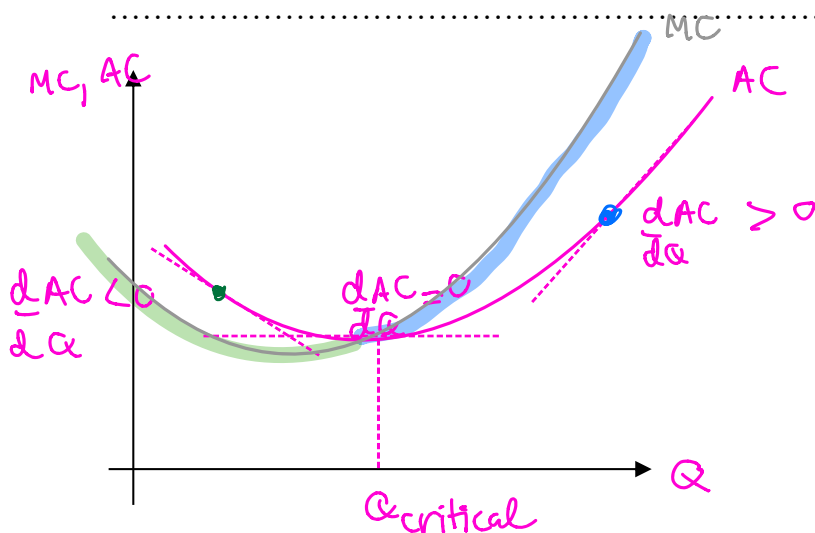
$$\textcircled{1} \text{ At minimum of AC} \Rightarrow \frac{dAC}{dQ} = 0 \rightarrow MC = AC$$

$$\textcircled{2} Q < Q_{\text{critical}} \Rightarrow \frac{dAC}{dQ} < 0 \rightarrow MC - AC < 0$$

$$MC < AC$$

$$\textcircled{3} Q > Q_{\text{critical}} \Rightarrow \frac{dAC}{dQ} > 0 \rightarrow MC - AC > 0$$

$$MC > AC$$



How much does total cost increase if labor in production increases?

$$TCC(Q) \quad Q(L)$$

$$TC(Q(L))$$

$$L \Delta \rightarrow Q \Delta \rightarrow TC \Delta$$

by Chain rule

$$\frac{dTC}{dL} = \frac{dTC}{dQ} \cdot \frac{dQ}{dL} = MC \times MP_L = MFC$$

"Marginal Factor Cost"

TR, AR, MR

$$Q^d = a - bP$$

Demand fⁿ

$$TR = P_{mkt} Q$$

$$P = \frac{a}{b} - \frac{1}{b} Q^d$$

Inverse demand fⁿ

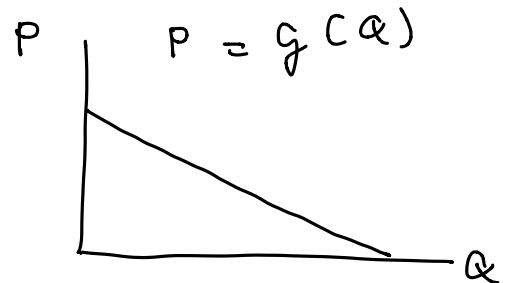
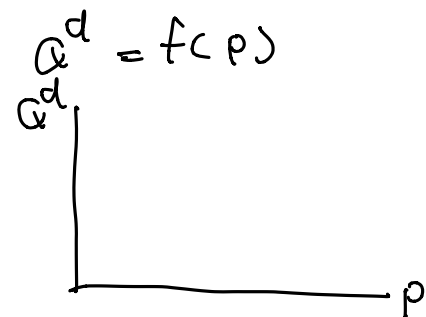
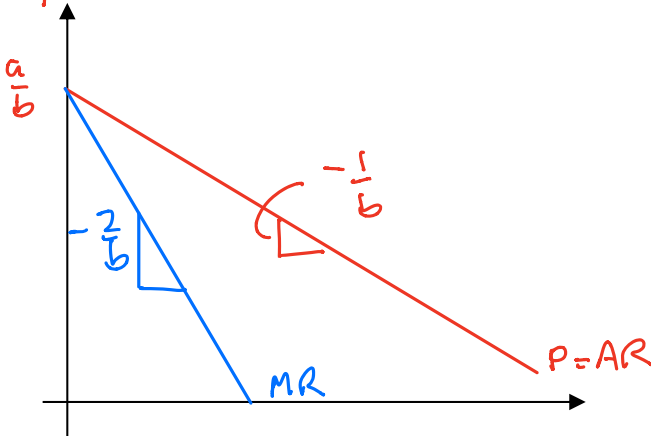
In a monopoly mkt.:

$$TR = PQ = \left(\frac{a}{b} - \frac{1}{b} Q \right) Q = \frac{aQ}{b} - \frac{1}{b} Q^2$$

$$AR = \frac{TR}{Q} = \frac{a}{b} - \frac{1}{b} Q \quad ; \text{ notice } ; P = AR$$

$$MR = \frac{dTR}{dQ} = \frac{a}{b} - \frac{2}{b} Q \quad ; \text{ notice } ; |MR'| = 2 |AR'| = 2 \left(\frac{1}{b} \right)$$

P, AR, MR



How much does total revenue increase if labor in production increases?

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H.W. Let $P = 25 - 0.1Q$, $Q = 5L$, find MRP

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↗ TP, AP_L, MP_L

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↗ **Demand** (1.) $Q^d = a - bP$; $a, b > 0$

I. $\frac{dQ^d}{dP}$

II. $\frac{d^2Q^d}{dP^2}$

(2.) $Q^d = \frac{a}{P}$

The graphs are:

↗ $\frac{dy}{dx}$ and marginality

- Utility

$$\frac{dU}{dx}$$

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$$\frac{d^2U}{dx^2}$$

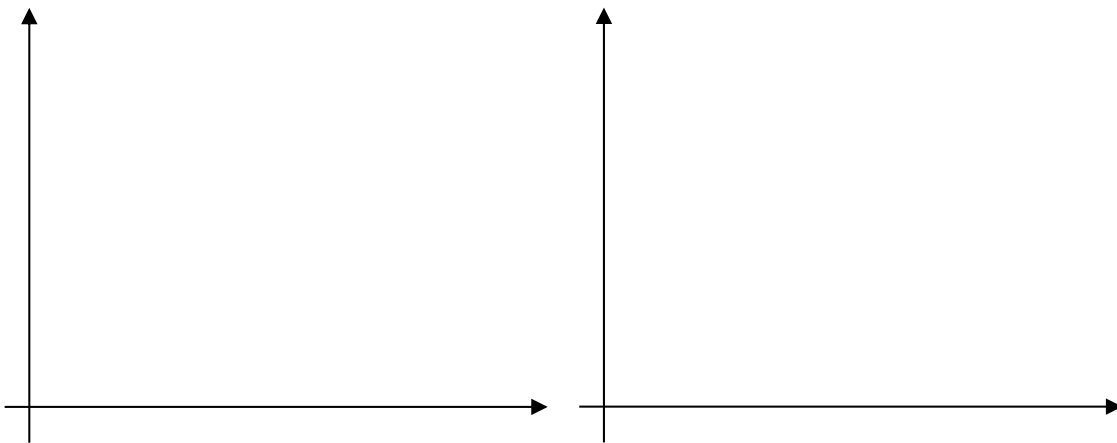
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H.W.

- Find marginal utility when $U = 5x^{1/2} + 100$
- Find marginal cost when $TC = 5Q^2 + 3Q + 200$
- Find marginal revenue when $TR = 25Q$

↗ $\frac{dy}{dx}$ and Elasticity

- $\frac{dy}{dx}$ and price elasticity of demand: law of demand

Demand (1.) $Q^d = 250 - 10P$

(2.) $Q^d = 50P^{-1/4}$

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- $\frac{dy}{dx}$ and income elasticity of demand: inferior vs. normal good

$Q^d = f(I)$

$E_I = \frac{dQ^d}{dI} \cdot \frac{I}{Q^d}$

$E_I < 0$

$E_I > 0$

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- $\frac{dy}{dx}$ and cross price elasticity of demand: substitute vs. complementary goods

$Q_a^d = f(P_b)$

$E_C = \frac{dQ_a^d}{dP_b} \cdot \frac{P_b}{Q_a^d}$

$E_C < 0$

$E_C > 0$

- $\frac{dy}{dx}$ and law of supply

$$E_s = \frac{dQ^s}{dP} \cdot \frac{P}{Q^s}$$

- $\frac{dy}{dx}$ and output elasticity of labor

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