Lecture on introductory Mathematical Economics

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Chapter 1:
Nature of economics analysis and Element of mathematical economics model

1.1 Structure of economic analysis

- How do we study economics?
  - Model-based analysis
- What is model?
  - Model is an *abstract* form of the problem that we are interested in.
  - The problem that we (economists) are interested in aims at getting us some insights, helping us understand something about how the world/economy works.
  - Typically, all the results in a certain model hinges on the *assumptions* used in the analysis.

- Purpose of economics model: Developed to answer three broad classes of analytical questions;
  - *Static equilibrium analysis*
  - *Comparative static analysis*
  - *Dynamic analysis*

Example 1.A: ***The Price determination model***

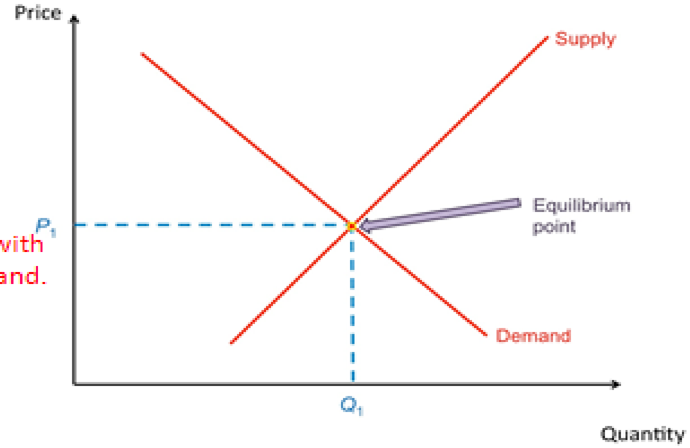
Question: At what price and quantity of output, market will trade?
- We know that there must be buyers and sellers in the market.
- To understand their trading decision, economists model the interaction of the two trading parties in the market using the basic **demand and supply model**.
- Conventionally, behavioral assumptions are that
  - Demand: Given all other factors, quantity demanded is negatively related to price
    - Downward sloping curve
  - Supply: Given all other factors, quantity supplied is positively related to price.
    - Upward sloping curve

Structure of economic analysis: Static equilibrium analysis

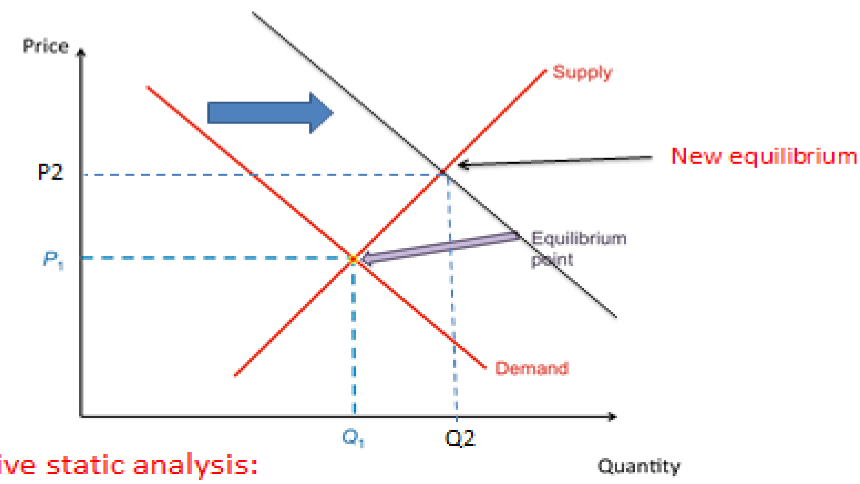
Static analysis:

- Draw demand/supply onto p-q diagram

- Equilibrium occurs when price clears the market. That is, demand matches with supply, aka no excess demand.

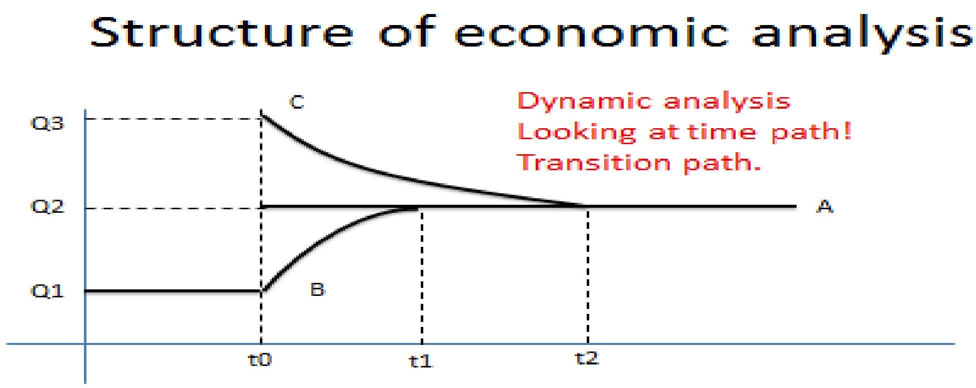


Structure of economic analysis: Comparative static analysis



Comparative static analysis:
For example, what if income increases?

Dynamic analysis



Where we are headed in this semester is to formalize all these analytical questions by using *mathematical tools*.

1.2 Elements of mathematical economic model

- Typically, any *mathematical models* involve with the following ingredients.
    - Variables
        - Quantity or amount of some objects whose value can be changed.
        - For example, Y is a variable that represents "your income in a day". X is another variable that represents how many hours you have worked in a day. (The value of both Y and X can be changed.)
    - Equation
        - An equation describes the relationship of variables, at the least two.
        - How Y and X can be linked together is summarized by an equation.
        - For example, $Y = 9X$ if hourly wage (w) is \$9.

- Nature of mathematical economic models

In general, mathematical economic models comprise of set of equations (more than one equation), each of which describes the interconnection among sets of variables. Mathematically speaking, the model has a feature of what so called *system of simultaneous equations*.

$$\begin{aligned} a_{11}x_1 + \ & a_{12}x_2 + \cdots + \ a_{1n}x_n = b_1 \\ a_{21}x_1 + \ & a_{22}x_2 + \cdots + \ a_{2n}x_n = b_2 \\ \vdots \qquad & \ \ \vdots \qquad\qquad\quad \vdots \qquad\ \ \vdots \\ a_{m1}x_1 + \ & a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{aligned}$$

The example above is to represent a model in the ***linear system***. Model can take the ***non-linear form***.

Sometimes, mathematical economic model takes the form of set of optimality conditions derived from optimizing agent problem. Optimality condition could be a single equation or system of equations. But in any cases, the condition describes the relationship between the variables that we aim to understand their behavior and the determinant variables.

To understand common structure and basic terminologies used in mathematical economics, consider *Example 1.B* which is the reformulation of the basic market equilibrium model using mathematical approach.

Example 1.B: Consider market equilibrium model as an example.

Equation 1: $Q_d = a - bP - cY$;

Equation 2: $Q_s = d + eP - fT$;

Equation 3: $Q_d = Q_s$

Notations defined as follows:

- Q_d = quantity demanded, Q_s = quantity supplied, P = price,
- Y = income, T= temperature ➔ variables

- (a, b, c, d, e, f) = *coefficients* or *parameters* that are all positive

Treatment of the type of equations:

- Behavioral equations
- Equilibrium conditions

Treatment of the variables:

- Single equation approach:
  - Independent variable (Explanatory)
  - Dependent variable
- System-based approach
  - Endogenous variable:______________________________
  - Exogenous variable:________________________________

1.3.1) Static Equilibrium analysis

From example 1.B, the most common goal of studying the model is to study the equilibrium analysis, i.e. to solve for the solution of the simultaneous equations. As the exogenous variables are treated as given, solution to the system of equation would represent the *endogenous equilibrium solution*. Technically speaking, endogenous equilibrium solution is called the *reduced-form equation*.

Reduced-form equation is the mathematical function that describes the behavior of the endogenous equilibrium solution. *Typically, the function would depend on (i) exogenous variables and (ii) parameters.*

- Type of variables treated/distinguished in the model
  - Exogenous variables = treated as "*given*" in the model. Their values are given as known thing, and *will not be solved* within the model.

- Endogenous variable = set of variables that will be solved for.

Example 1.C: National income model

A very basic/simple version of Keynesian cross model
- Consumption function: $C = \mathrm{C}(\mathrm{Y}, \mathrm{a}, \mathrm{b})$
- Investment equation: $I = I_0$
- Government equation: $G = G_0$
- Equilibrium condition: $Y = C + I + G$

Variable Lists:
Y = income, C = consumption
I = investment and G = government expenditure

- ***Static equilibrium analysis***: characterizing the equilibrium solution of the model by solving for a solution of the system of equations. By the definition, the reduced-form solution takes the following *generic* representation.

$$Y^* = f(I_0, G_0, a, b)$$

$$C^* = g(I_0, G_0, a, b)$$

- If we assume the function form, the reduced-from solution would havean analytical (specific) solution which is specific to the assuming conditions on the behavioral equation.
    - For example, we might assume that $\mathrm{C}(\mathrm{Y}, \mathrm{a}, \mathrm{b}) = \mathrm{a} + \mathrm{bY}$

Keywords: *Reduced-form equation, generic solution, specific solution*

1.3.2) Comparative Static analysis:

Having solved for the equilibrium solution, what economists usually ask is what would happen to the equilibrium if something, previously assumed to be fixed, has changed.

Example 1.C (cont.): National income model

- From the example 1.B, it is straightforward to solve for all the endogenous equilibrium solutions, Y^*, C^*.

- Numerically, if a = 1, I_0 = \$1, G_0 = \$1 and b = 0.5, this yields us,

Question: What if G is now changed to \$2, how big is the change in Y* and C*?

Answer:

New Y* =_______________ ➔

A $1 increase in G causes an increase in Y* by _________

In the later part, we will try to figure out the level of multiplier when we don't assign any numerical values of variables.

- The idea is simple. *We just apply the derivative method to the equilibrium solution function*
- That is we calculate the value of $\frac{\partial Y^*}{\partial I_0}$, $\frac{\partial Y^*}{\partial G_0}$, $\frac{\partial Y^*}{\partial a}$, $\frac{\partial Y^*}{\partial b}$.

Chapter 2: Function and relation

2.1 Function and relation in economics

- Relation: mapping between two sets of variables.
  - One-to-one
  - Many-to-one
  - One-to-many

- Function is two special classes of relation
  - One-to-one
  - Many-to-one

Domain and Range

To properly define any mathematical functions, the function must be associated with domain/range. See example 2.A

Example 2.A : $y = \sqrt{x} - 3$

Domain for x __________________________

Range for y __________________________

<u>Generic function *and* Economic function:</u>

Having applied math into economic function, we need to be very careful about the restricted sets of domains and ranges that makes the function *<u>economically meaningful</u>*. See example 2.B below!

Example 2.B: Does the domain set for x continue to be the same as before if we now are using X to represent price, and y to represent quantity? Discuss, why or why not!

2.2 Types of function commonly used in Economics

2.2.1) Linear function

A linear function takes the form:

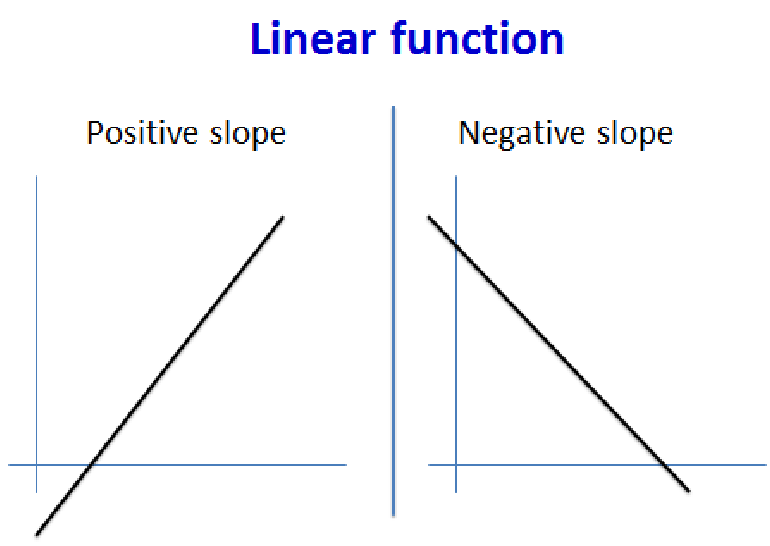
$$y = a + mx$$

$$a = intercept\ \ on\ y - axis\ (x = 0,\ y = a)$$

$$m = slope$$

What is the slope?

- A change in "y" with respect to a change in "x".
- Standard notation refers to $\frac{\Delta y}{\Delta x}$ as the slope of a function.
- Generally, slope can be varied with respect to changing values of X.
- But for the linear function, it's not. $\frac{\Delta y}{\Delta x} = m$ ➔ constant!
- Graphical illustration of a linear function depends on the value of slope, i.e. m.



Note: With the constant slope, suppose we know a line goes through two points (A and B), we can derive the equation representing that line, using the two-step procedure.

- Steps 1: Deriving slope

- Steps 2: Solve for "a" by using slope and one of the two points given.

See **example 2.C** below

Example 2.C: Suppose a research has a priori belief that the relationship between "q" and "p" is linear. From the survey, the research only knows that when p is equal to 1, q is equal to 4. Meanwhile, when p is equal to 5, q is then equal to 0. Find the equation of the line that represents the p-q relationship? What does the relationship represent?

Example 2.D : Suppose $Q = 20 - P - Y$ as a market demand equation, where Q is quantity, P is price and Y is income.

- Find the domain set of demand.

- Plot the demand curve for Y = 5, and compare with the demand curve for Y = 10.

Example 2.E: Suppose $Q_x = 10 - P_x + bP_y$ where

Q_x is quantity demanded for x.

P_x is price of good x.

P_y is price of good y.

What could be summarized as the relationship between good x and good y under which "b" is positive, zero, and negative?

Conclusion

Given $Q_x = k - aP_x + bP_y$, we can summarize that if

- b = positive ➔ an increase in P_y ➔ Q_x increases ➔ *substitute product*
  - I-phone vs Samsung
- b = negative➔ an increase in P_y ➔ Q_x decrease ➔ *complement product*
  - Coffee and tea
- b = 0 ➔ an increase in P_y ➔ Q_x stay the same ➔ *neither substitute nor complement*
  - toilet paper and Ferrari

Slope v.s. elasticity

General notation:

Let ε_x^y be the elasticity of "y" with respect to "x". The elasticity is then defined as follow

$$\frac{\%\Delta y}{\%\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} * \frac{x_0}{y_0} = \frac{\Delta y}{\Delta x} * \frac{x_0}{y_0} = Slope * \frac{x_0}{y_0}$$

- Elasticity depends on the starting value. (starting point matters!)

Example 2.F: *Elasticity of demand*

Suppose we know any two points on the demand curve, namely $A(Q_0 = 4, P_0 = 1)$ and $B(Q_1 = 1, P_1 = 5)$ where A is the initial point and B is the terminal point

Elasticity of demand of demand is then defined as

$$\frac{\%\Delta Q}{\%\Delta P} = \frac{Q_1 - Q_0}{P_1 - P_0} * \frac{P_0}{Q_0} = \frac{\Delta Q}{\Delta P} * \frac{P_0}{Q_0}$$

More about the elasticity of demand

- Own-price elasticity of demand is always less than or equal to zero! (why?)
    - Negative sign only implies that negative relationship between price and quantity demand.
    - Absolute number of the figure tells us about the degree of sensitivity of quantity demand to price adjustment.

- Conventionally, we classify the case for elasticity of demand into three cases.

$$\left|\frac{\%\Delta Q}{\%\Delta P}\right| : \begin{array}{lcc} >1 & ; & elastic \\ =1 & ; & unit\ elastic \\ <1 & ; & inelastic \end{array}$$

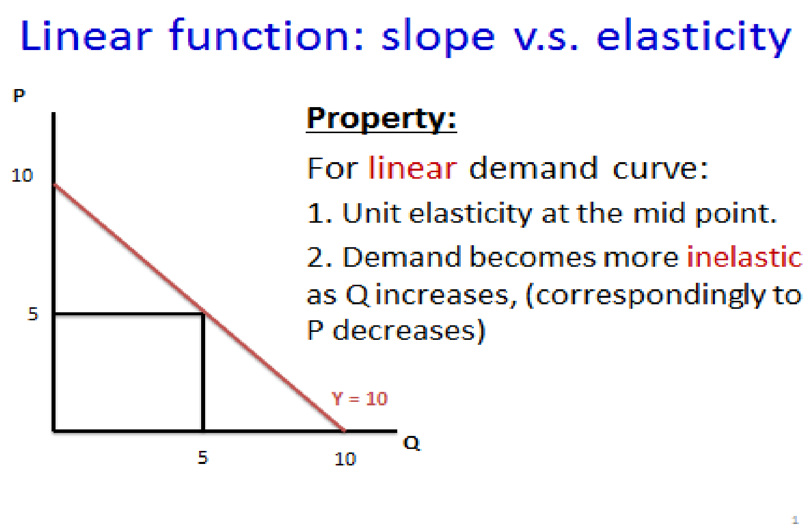
Discuss about some important implications of elasticity for practical use.

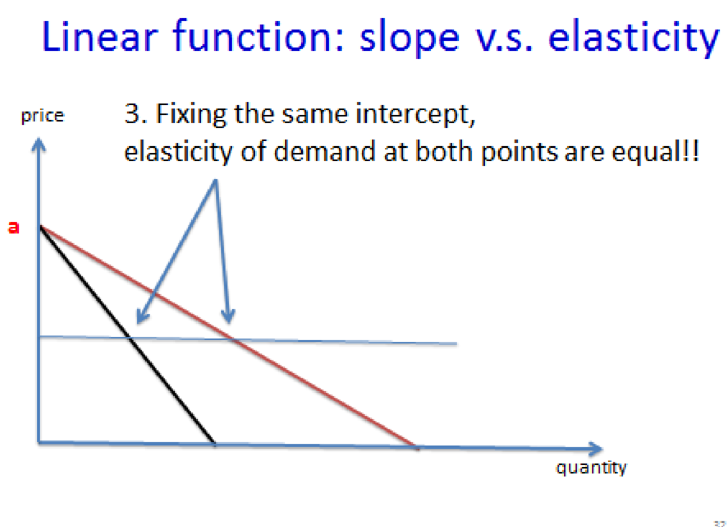
- Boosting revenue

- Optimal commodity taxation

Example 2.G: Elasticity of linear demand curve and some important properties

Suppose that $p = 10 - Q$, derive the formula for the elasticity of demand





Exercise 2.A:

2.A.1) Given a demand function by $p = a - bQ$, derive the formula for the elasticity of demand, and show that the third property holds

2.A.2) Given the market supply $p = c + dQ$ where $d \geq 0$, show that

(i) elasticity of supply is always greater than 1 if c > 0,

(ii) elasticity of supply is always eqaul to 1 if c = 0,

(iii) elasticity of supply is always less to 1 if c < 0.

Example 2.H: Aggregate expenditure function

Consider a simple macroeconomic model where

$$C = 100 + 0.7Y$$

$$I = 200 + 0.1Y - 100 * r$$

- Derive the saving function and plot both consumption and saving function in the same figure

- Derive the Aggregate Expenditure function. How does the aggregate expenditure behave over the course of changes in the interest rate?

2.2.2 Non-linear functions

Drawbacks of the linear representation.

- Constant slope is a big deal of the problem.
- SES Panel survey shows that the rich tend to consume less out of each additional unit of income they have earned.
  - Thus, MPC should be decreasing as income increases.
- So, neither the slope of consumption function should be constant nor the form of consumption function could be truly represented by a linear equation. (In fact, the consumption function should be concave!)
- Two broad classes of non-linear functions commonly used.
  - Polynomial function
  - Exponential/Logarithm function

2.2.2a) Polynomial function

Polynomial takes the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

N = 1 ➔ linear ➔ straight line

N = 2 ➔ Quadratic function ➔ Parabola

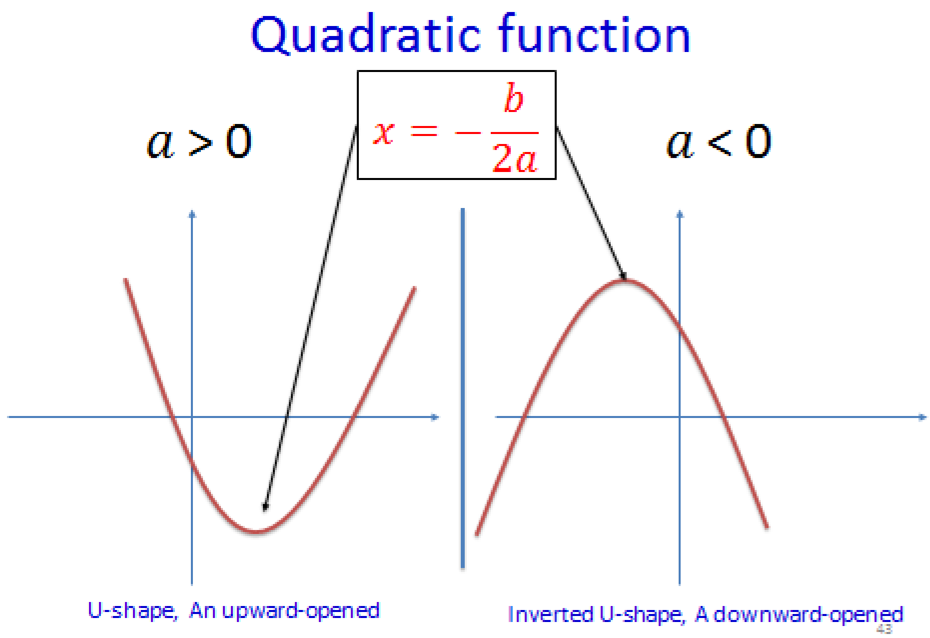
N = 3 ➔ Cubic function ➔ Don't know, but who care

The Quadratic function

Consider the polynomial function with degree 2.

- Let's write the function in the form that you are used to.

$$y = ax^2 + bx + c$$



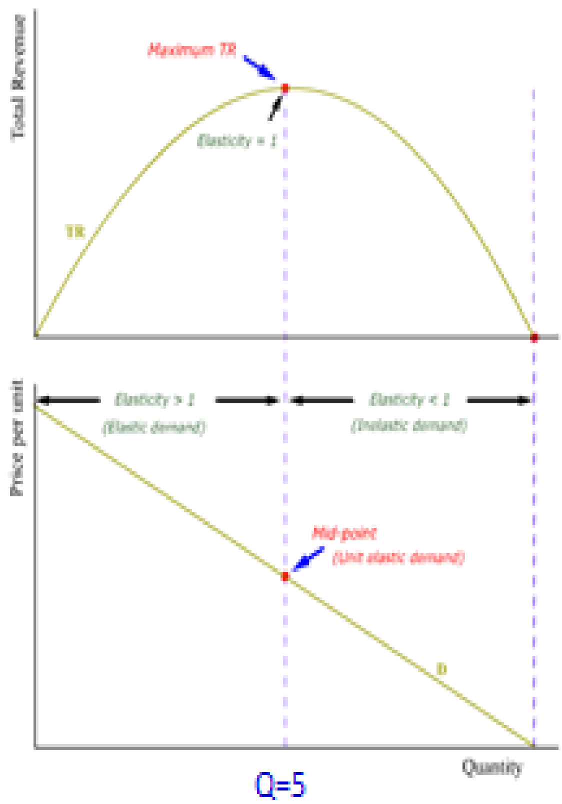
Example 2.I: A monopolist firm faces the market demand given by $P = 10 - Q$. Consider the following questions if the cost function $C(Q) = 4Q$.

- What is the revenue-maximizing level of output?

- What is the break-even output?

- What is the profit-maximizing level of output?

Quadratic function: revenue function/break even analysis



$$\left|\frac{\%\Delta Q}{\%\Delta P}\right| : \begin{array}{lll} > 1 & ; & \textit{elastic} \\ = 1 & ; & \textit{unit elastic} \\ < 1 & ; & \textit{inelastic} \end{array}$$

Revenue maximizing output:
unit elasticity

Profit-maximizing output:
under region with **elastic** demand

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2.2.3b) Exponential and Logarithm

Another two functions commonly used in economics analysis are exponential and logarithm. The two functions are inverse function of each other.

Exponential : $y = a^x$

Logarithm : $y = \log_a x$

Example 2.J: Bond pricing formula

A Two-year bond offers a stream of payment for \$10 in every single year until it is matured. At the maturity date, this bond also delivers the principal amount at \$100. If market interest rate is r% per annual, what would be the fair price of the two-year bond?

Chapter 3: Equilibrium *and* Comparative static analysis: Linear model

3.1 Linear economics model

What we have been discussing so far centers on

- How to characterize relationship of economic variables using mathematical function/equation.
- Properly define the function/equation in economics analysis.

What we are going next is to

- Write down some basic linear economic models (in mathematical form)
  - Set of system of equations.
  - Interlink among variables in the model.
- Solve for the solution of the model => *Static analysis*
  - (Simultaneous) Solution to the system of equations
- Analyze behavior of the endogenous equilibrium solutions => *Comparative static analysis*

List of the economic models that we will be taking about include

- Micro-market equilibrium model
  - Single market equilibrium (Partial analysis)
  - Multi-market equilibrium (General analysis)
- Macroeconomic model
  - Keynesian cross
  - IS-LM model

3.2 System of equations

- Comprises of *N-unknown* variables **and** *M-equations*.
  - Equations could be linear, nonlinear, or mixed.
  - But we now focus on linear system.

- Set of N-unknown variables that simultaneously satisfies all the M equations, at the same time, is called solution of the system.

- For now, we will only focus on
  - 2 x 2 case
  - 3 x 3 case

- These two cases are solvable by *pencil and paper*.

- For any larger scale of the system of equations, we have to apply matrix algebra to solve for the solution. (In the later chapter)

Example 3.A: Solving for the solution of the system of equations

$X + Y = 7$

$3X - 6y = -15;$

Example 3.B: Distinguishing the type of solution

- System 1: $2x + 3y = 7$ and $x - y = 5$.
- System 2: $y = -2x + 6$ and $4x + 2y = 12$.
- System 3: $x + 2y = 14$ and $3x + 6y = 8$

3.3 Market Equilibrium model

- What is the market equilibrium model?
  - The model is used to predict equilibrium price and output.
    - At what price, people trade?
    - How much do they trade?
  - Two concepts: ***partial equilibrium (PE) vs general equilibrium (GE)***
    - PE: One particular market, ignore the rest.
    - GE: All the interlinked markets are taken into consideration at the same time.

Example 3.C Compare and contrast between PE and GE model

Partial	General
$Q_d^x = a - bP_x + cP_y$	$Q_d^x = a - bP_x + cP_y$
$Q_s^x = d + eP_x$	$Q_s^x = d + eP_x$
P_y are given exogenously	$Q_d^y = f - gP_y + hP_x$
	$Q_s^y = j + kP_y$
	(P_x, P_y) are solved for together.

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3.3.1) Partial equilibrium model

Example 3.D Suppose that *market* demand and supply equation can be given by the following two equations:

$$Q_d = 30 - 5Y - 3P$$

and $P = 2 + Q_s$

a. Graph the demand and supply curve for Y = 2

b. Is the product normal/inferior goods?

c. At what level of income, does the demand function coincide with the equation $Q_d = 10 - 3P$?

d. Find the market equilibrium for quantity of output and price, using the demand equation given in "c".

e. **What happen to equilibrium output/price when income is $1 higher than the level associated to that in "c"? Interpret your result.

Caution: Equilibrium notion requires the information of the *market* demand and *market* supply. Mostly, this information needs to be derived from the given individual demand and supply.

Definition:

1. Market demand is the total quantity demanded of each individual (consumer/buyer) combined.

$$Q^d = \sum_{i=1}^{N} Q_i^d$$

where Q^d is total quantity demanded at the market level.
Q_i^d is the quantity demanded for i-th individual.

2. Market supply is the total quantity supplied of each individual (firm/seller) combined.

$$Q^S = \sum_{i=1}^{M} Q_i^S$$

where Q^S is total quantity demanded at the market level.
Q_i^S is the quantity demanded for i-th firm.

Two methods for the aggregation!

- Direction summation approach
  - Apply basic sigma operation
- Horizontal summation approach
  - Deriving market demand curve from individual demand curves
  - Find the mathematical equation that represent the derived market curve

See Example 3.E below

Example 3.E: Market demand

Consumer 1: $Q_1^d = 3 - P$

Consumer 2: $Q_2^d = 2 - P$

Direct Summation:

Deriving market demand curve from individual demand curves.

What is the mathematical equation that represents the market demand curve when using the horizontal summation method? Derive the equation/function.

Example 3.F: (market supply):

A market has two firms, A and B. Each has the supply equation given by,

Firm A: P = 15 + Qa

Firm B: Qb = 0.5P - 5

Derive the market supply equation

Example 3.G: Solve for the market equilibrium using the information in **Example 3.E** and **Example 3.F.** Justify your answer!

Some extensions: Government interventions

Taxation

- The first type of government intervention is *taxation*
- Micro-market model allows us to understand the impact of commodity tax.
- Basically, there are two types of commodity tax.
  - Unit tax ➔ $t per each unit of production.
  - Ad-valorem tax ➔ t% of value.
- Let's start from the impact of unit tax. Then, we will come back to the analysis on Ad-valorem tax later.
- The purpose of this part is to show:
  - A rigorous way to analyze the impact taxation on price and allocation in the market equilibrium.
  - Showing that the implication of tax on the welfare, and distribution of welfare cost among stakeholders in the market.

Impact of the unit taxation

The basic set up of linear market model typically takes the form:

Demand: $p \quad = a - bQ^d \quad ; \; a \geq 0, \quad b \geq 0.$

Supply : $p \quad = c + dQ^s \quad ; \; d \geq 0.$

- Equilibrium can be then solved by imposing the market clearing condition, i.e. $Q^d = Q^s$

When the government imposes tax, the followings happen.

1. With the imposition of tax, *price that consumer's paying will no longer be equal to price that producer's receiving*.
   - The single "p" concept that underlies the basic set up of linear market model would no longer be sensible!
   - So, to analyze the impact of taxation, we need to make it clear from now that
     - $p^s = price\ that\ producer\ gets$
     - $p^d = price\ that\ consumer\ pays$
     - $p^s \neq p^d$ ➔ price gap *or* price wedges

Behavior of buyers and suppliers would focus at their perceived prices.

That is, the market model needs to be rewritten in the following way.

Demand: $p^d = a - bQ^d \quad ; \; a \geq 0, \quad b \geq 0.$

Supply : $p^s = c + dQ^s \quad ; \; d \geq 0.$

2. With the imposition of tax, equilibrium allocation would be affected!

   - VERY SIMPLE, we solve for equilibrium by imposing that $Q^d = Q^s$.
   - BUT!!!!

Example 3.H Can we solve for the equilibrium by simply imposing the only market clearing condition?

Solvable? If not, what else do we need?

Example 3.I Suppose that demand/supply equation can be given by

$$P \quad = 80 - 3Q^d$$

$$P = 10 + 2Q^s$$

- Find the market equilibrium.

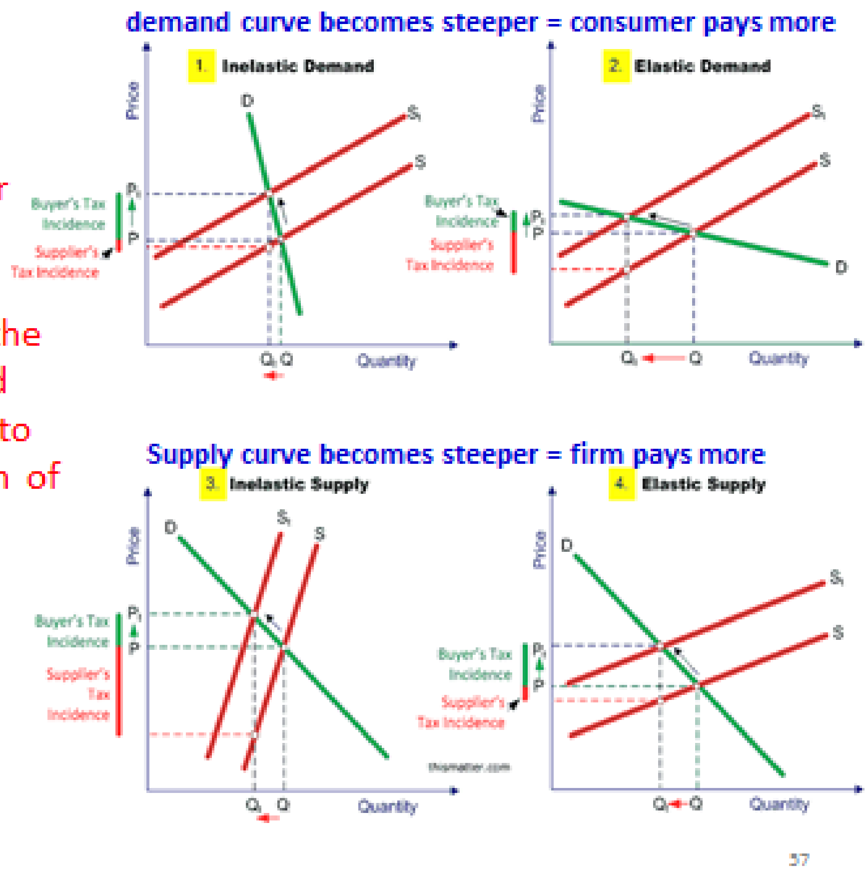
- Suppose that government imposes tax on producer equal to $5 per unit, determine market equilibrium under taxation.

- How much is the tax revenue that the government can collect in the equilibrium?

- Calculate the tax burden.

Excess burden/Tax incidence:

Main Idea: Whoever cannot adjust themselves instantaneously to the new situation would unfortunately have to bear a larger portion of the tax burden.



Example 3.J: Excess burden *formula under linear model*

Demand: $p^d = a - bQ^d \quad ; \ a \geq 0, \quad b \leq 0.$

Supply : $p^s = c + dQ^s \quad ; \ d \geq 0.$

- Solve for quantity and prices equilibrium when the unit tax is imposed. Analyze the result

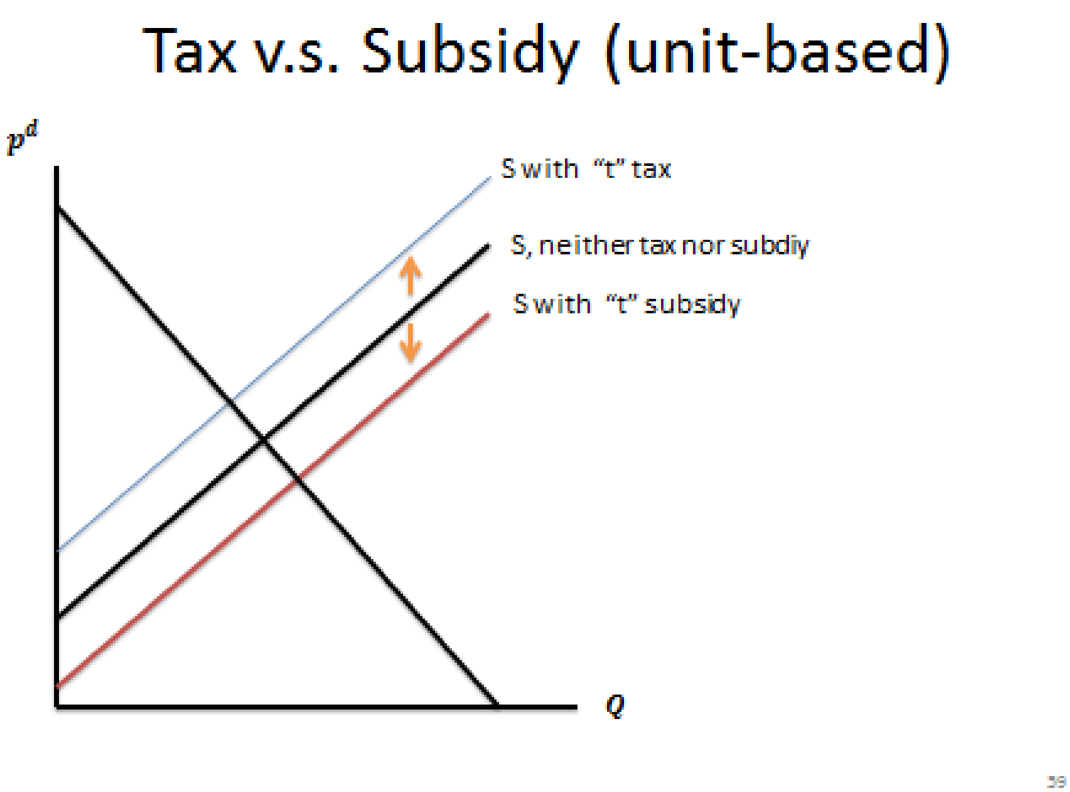
- Derive the excess burden formula for buyers and sellers

Unit-subsidy?

- Do we have to start everything from the beginning to derive the implications of subsidy program for market equilibrium?

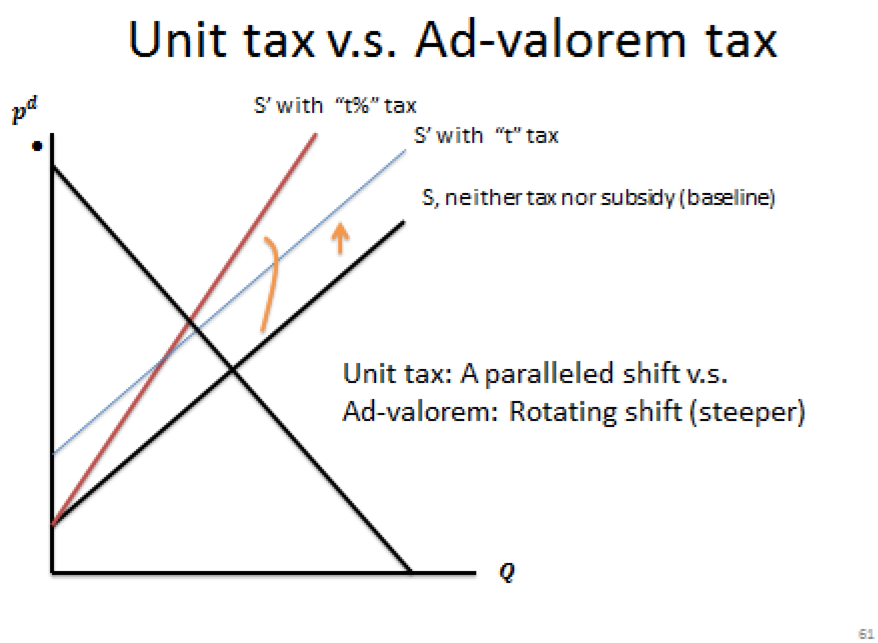
    - No, the analytical result for the impact of subsidy would have been the same, because we can treat subsidy as a negative tax. “t” is negative number!

    - The burden concept would be changed into the benefit received, i.e. who’s getting more benefit.



Ad-valorem tax

- Tax by t% of price!. An example is the value-added tax.
- *Most* Ad-valorem taxes are imposed at the point of purchase, i.e. imposing on consumers
- If the tax is imposed on consumer, it would be that: $p^d = p^s\left(1 + \frac{t}{100}\right)$.



Example 3.K Price control and Welfare

Consider the market for apartment rentals in Chicago. The price of rent is determined by the following system of equations.

Demand: $p = -2q_d + 160$

Supply: $p = q_s + 10$

- What is the equilibrium price and quantity in the market for apartment rentals?

- Suppose the government tries to control the rent prices through a price ceiling of $40. Discuss the implication of this policy. Is there any deadweight loss?

3.3.2) Multi-market model (General equilibrium model)

- So far, we have only analyzed the market equilibrium concept when we assume that there is only single market in the economy.

- In fact, the economy comprises of several markets.

- Equilibrium concept that we use so far can be extended to the more realistic environment where we have N markets.

Example 3.K Let demand and supply of market for goods A and B be

$$D_a = 20 - 3Pa - 2Pb$$

$$S_a = 12 + 2Pa + 5Pb$$

$$D_b = 4 - Pa - 3Pb$$

$$S_b = -1 + 2Pa + 4Pb$$

- State equilibrium conditions of this multimarket model.

- Find the equilibrium price and quantity of the market for goods A and the equilibrium price and quantity of the market for goods B.

3.4) Simple macroeconomics models

- **Keynesian cross model (partial** equilibrium **model)**
    - Analyze equilibrium determined of output in the goods market.
    - Treating financial sector as given, i.e treating market interest rate as given!
        - Making sense under the liquidity trap environment
    - Implications: emphasizing at the role of fiscal policy

- **IS-LM model (General** equilibrium **model)**
    - Endogenize the market interest rate.
    - Introduce the feedback/interactive effect between financial market and goods market.
    - Implications: emphasizing at the role of both fiscal and monetary policy.

Where we are headed!

- Solving for equilibrium solution of endogenous variables.
- Some comparative static analysis: analyze behavior of endogenous variables.
- Policy issues
    - *Multipliers*
    - *Crowding-out effects*
    - *Policy effectiveness*

3.4.1) The basic workhorse for the Keynesian cross model

From example 1.B, the simplest workhorse for the Keynesian cross model is the one that assumes a closed economy with no government sector. The model can be explained by 3 equations followed.

$Y = C + I$ ➔ equilibrium condition

$C = \mathrm{C(Y)} = \mathrm{a + bY}$ ➔ behavior of consumption

$I = I_0$ ➔ exogenous investment (take it as given.)

- Exogenous: I_0.
- Endogenous: C and Y.
- Parameters: a (autonomous consumption), b (mpc)

Static analysis: Equilibrium solution of endogenous variables takes the form

$$\mathrm{Y}^* = \frac{\mathrm{a + I_0}}{1 - \mathrm{b}}$$

$$\mathrm{C}^* = \mathrm{a + b}\left(\frac{\mathrm{a + I_0}}{1 - \mathrm{b}}\right)$$

Sensitivity analysis: suppose that $\mathrm{I_0} \to \mathrm{I_0'}$, what would happen to the equilibrium?

$$\Delta Y^* = Y^{*\prime} - Y^* = \frac{1}{1-b} * (I_0' - I_0)$$

$$\Delta C^* = C^{*\prime} - C^* = \frac{b}{1-b} * (I_0' - I_0)$$

Extensions: *Simplicity and reality*

One can enrich the basic model in several directions; this allows us to yield the model that betters reflecting how the world/economy actually works. An exhaustive list of possible extension includes:

- Endogenous investment
  - Investment also depends on level of income, i.e. I(y)
  - Income-accelerator framework, i.e. $I(\Delta y)$ (Dynamic)

- Government sector
  - Government spending
    - Typically treated as an exogenous variable.
    - Alternatively, G might be tied to the level of GDP.
    - Automatic stabilizer, i.e. counter-cyclical government spending rule.
  - Taxation
    - Consumption depends on disposable income, i.e. C(Y-T)
    - Income tax, i.e. T(y)

- Effect of interest rate on private spending
  - Consumption and investment could also depend on interest rate, i.e. C(y,r) and I(y,r)
    - This extension reflects the idea that cost of borrowing matters to both households' and firms' spending.
  - Also, introducing the role of interest rate allows for the channel of linkage between real and financial sector!

- Open economy
  - Equilibrium condition must include net export, i.e. X – M.
  - Need to model the behavior of "X" and "M".

Example 3.L: A full-version of Keynesian cross model

Equilibrium (1)	$Y = C + I + G$
Consumption function (2)	$C = a + bY_d$
Disposable income (3)	$Y_d = Y - T$
Tax rule (4)	$T = T_0 + t * Y$
Investment function (5)	$I = I_0 - I_1 r$
Government spending (6)	$G = G_0$

- Solve for the equilibrium output function?

- Multiplier of autonomous expenditures

3.4.2) The IS-LM model

- So far, we have seen a simple Keynesian cross model that "r" is exogenously given.
  - An increase in "r" reduces aggregate expenditure, and hence the level of income in the equilibrium.
- Is the assumption that "r" an exogenous variable sensible? Not really!..
- IS-LM model is an extension of the Keynesian cross.
  - The model *endogenizes* the value of interest rate by modeling how interest rate is determined in the equilibrium.
- Conceptually, interest rate is determined from the equilibrium in money market, where money demand is equal to money supply.
- In economics, we typically assume that money demand depends on (i) income and (ii) interest rate.
- Since money demand also depends on income, equilibrium level of interest rate in the money market also depends on the level of income.
- As a result, income and interest rate will have to be simultaneously determining each other in the general equilibrium.

Money market modeling

Equilibrium condition $M^d = M^s$

Money demand equation $M^d = \mathrm{L}_0 + \mathrm{L}_1\mathrm{Y} - \mathrm{L}_2\mathrm{r}$

Money supply $M^s = M_0^s$

Characterizing equilibrium in the money market

- Putting everything into the equilibrium condition

$$L_0 + L_1 Y - L_2 r = M_0^s$$

- Suppose that "Y" is given for now. Write "r" in terms of "Y" and "M_0^s", we yield **the LM equation**,

$$r = \frac{1}{L_2}\left(L_0 + L_1 Y - M_0^s\right)$$

General Equilibrium in the IS-LM model

The equilibrium solution for Y* and r* simultaneously satisfy the following two equations

IS equation: $Y = \frac{1}{1-b(1-t)}[a - bT_0 + I_0 - I_1 r + G_0]$

LM equation: $r = \frac{1}{L_2}\left(L_0 + L_1 Y - M_0^s\right)$

Example 3.M Consider the following model equations that explain the behavior of goods and money market.

$C = 0.8(Y - T)$

$T = 1{,}000$

$I = 800 - 20r$

$G = 1{,}000$

$Y = C + I + G$

$M^d = 0.4Y - 40r$

$M^S = 1{,}200$

Write numerical formula for IS equation

Write numerical formula for LM equation

Solve for endogenous equilibrium solution of "Y", "C", "I" and "saving".

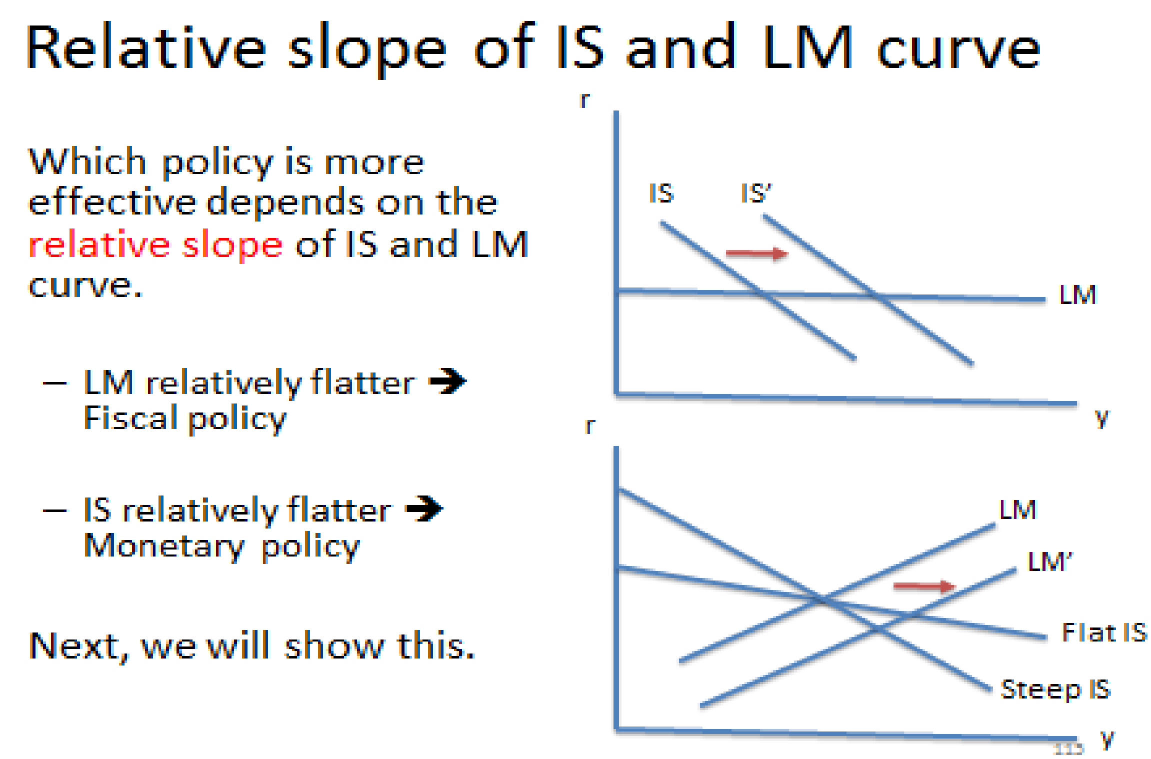
What happen to equilibrium "Y" when G increases by 200.

Back to when G is equal to 1000. What happen to equilibrium "Y" when money supply increases by 200.

Measuring in term of the change in output, which policy is more effective? Explain the reason.

The issue on Policy effectiveness!

- Compare the multipliers that arise under two different types of policy
- In general, government purchase is more effective than tax policy.
  - Is there a case against this conclusion? Think about it!
- What we have seen from EC212 are the following conclusions!
  - The flatter is the IS curve; the more effective is monetary policy in influencing output.
  - The flatter is the LM curve; the more effective is fiscal policy in influencing output.



Example 3.N: Using the IS-LM equation derived above, and show that

$$\frac{multiplier\ of\ M}{multiplier\ of\ G} = \frac{I_1}{L_2}$$

IS equation:

$$Y = \frac{1}{1 - b(1 - t)}[a - bT_0 + I_0 - I_1 r + G_0]$$

LM equation:

$$r = \frac{1}{L_2}\left(L_0 + L_1 Y - M_0^s\right)$$

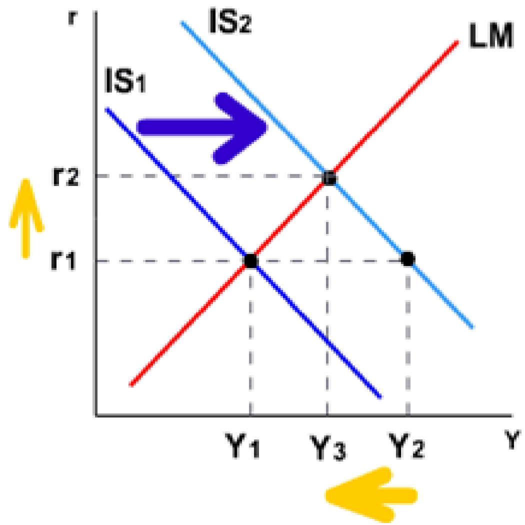
Economic Intuitions!

- **The flatter is the IS curve**; the more effective is monetary policy in influencing output.
  - IS curve is flatter when I_1 is high!
    - $Y = \frac{1}{1-b(1-t)}[a - bT_0 + I_0 - I_1 r + G_0]$
  - When I_1 is high, aggregate expenditure is highly sensitive to interest rate.
- **The flatter is the LM curve**; the more effective is fiscal policy in influencing output.
  - From the IS equation, $\mathrm{Y} = \frac{1}{1-b(1-t)}[a - bT_0 + I_0 - I_1 r + G_0]$
    - Mechanically, an increase in government expenditure is typically associated with an increase in income.
  - Following the initial increase in income, this would cause an increase in "r" which generates the crowding out effect of government expenditure, dampening the initial effect of G.
    - $r = \frac{1}{L_2}\left(L_0 + L_1 Y - M_0^s\right)$
  - Fiscal policy produces the strongest impact when it does not alter market interest rate, i.e. $L_2 = \infty$.
    - Under the less extreme case, fiscal policy is highly effective if the crowding effect is weak, i.e. when we have the high value L_2.
    - This is consistent with the situation when LM curve is flat.

Crowding out effect in the graphical illustration

Multiplier of G: Keynesian cross and IS-LM

- The crowding-out effect
  - Obviously, an increase in "G" causes an increase in "y".
  - If interest rate were kept fixed, effect of "G" would be very strong. (Y1Y2)
  - However, as "y" increases, interest rate tends to move along. (following the LM equation)
  - This would lower the investment, and thus reducing aggregate expenditure.
  - So, it partially offsets the effect of "G". (Y3Y2)
  - Income wouldn't be changing that much, comparing to when you don't have this feedback effect. (Y1Y3 vs Y1Y2)



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Chapter 4: Matrix algebra and its application

4.1) Notion of matrix and Basic operation

Matrix definition:

A matrix consisting of m horizontal rows and n vertical columns is called an ***m×n matrix*** or a ***matrix of size m×n***.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- For the entry a_{ij}, we call i the row subscript and j the column subscript.

Row Matrix or Row Vector

- *The case for which there is* single row *in the matrix, i.e. m = 1.*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

Column Matrix or Column Vector

- *The case for which there is* single column *in the matrix, i.e. n = 1.*

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \\ a_{m1} \end{bmatrix}$$

Example 4.A: Construct a matrix that has order of 3 x 4 and $a_{ij} = i+j$

Equality of matrices

Matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are equal if they have

(i) the same size and

(ii) $a_{ij} = b_{ij}$ for each i and j.

Transpose of a Matrix

A transpose matrix is denoted by A^T or A'

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow A^T \text{ is}$$

4.2) Some special matrices

Zero Matrix

- Can be called "Null Matrix"
- An m x n matrix with all entries or elements **are** zero.

Square Matrix

- A square matrix is a special type of matrix that has the same number of row as column, i.e. m = n.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Note that transpose of square matrix continue to be square matrix.

Diagonal Matrix

- A special type of square matrix whose off-diagonal elements are entirely zero.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

- Note that the case for which such matrix has all zero-entries is called *"Zero Matrix"*
- The case for which all main diagonal elements are unit is called *"Identity Matrix"* (To be discussed later!)

Triangular Matrix

- Upper Triangular
  - *A square matrix that contains only zero for all entries below its main diagonal.*
- Lower Triangular
  - *A square matrix that contains only zero for all entries above its main diagonal.*

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$$

4.3) Basic operations of matrix

Addition/Subtraction

Sum A + B is the $m \times n$ matrix obtained by adding corresponding entries of **A** and **B**.

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	(commutative property)
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	(associative property)
3. $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$	(identity property)

Example 4.B:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 7 & -2 \\ -6 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Example 4.C: Individual and aggregation

Demand for the consumers is

$$D_1 = \begin{bmatrix} 3 & 2 & 5 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 0 & 17 & 1 \end{bmatrix} \qquad D_3 = \begin{bmatrix} 4 & 6 & 12 \end{bmatrix}$$

For the industries is

$$D_C = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \qquad D_E = \begin{bmatrix} 20 & 0 & 8 \end{bmatrix} \qquad D_S = \begin{bmatrix} 30 & 5 & 0 \end{bmatrix}$$

What is the total demand for consumers and the industries?

4.4) Multiplication

4.4.1) Scalar multiplication

1. $k(\mathbf{A}+\mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
2. $(k+l)\mathbf{A} = k\mathbf{A} + l\mathbf{A}$
3. $k(l\mathbf{A}) = (kl)\mathbf{A}$
4. $0\mathbf{A} = \mathbf{O}$
5. $k\mathbf{O} = \mathbf{O}$

$$(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$$
$$(k\mathbf{A})^{\mathrm{T}} = k\mathbf{A}^{\mathrm{T}}$$

Example 4.D:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

Compute the followings

1. $3A^T + D$ 2. $(B-C)^T$
3. $2B^T - 3C^T$ 4. $2B + B^T$
5. $C^T - D$ 6. $\left(D - 2A^T\right)^T$

Exercise 4.A:

Find the value of x and y that makes the equality condition of the two matrices hold.

$$x\begin{bmatrix}2\\0\\3\end{bmatrix}+2\begin{bmatrix}-1\\0\\6\end{bmatrix}+y\begin{bmatrix}0\\2\\-3\end{bmatrix}=\begin{bmatrix}8\\4\\3x+12-3y\end{bmatrix}$$

4.4.2) Matrix multiplication

- Let A be m x n matrix and B be an p x q matrix.
- Then the product AB exists if n = p. (#Column A = #row B!)
- Given the necessary condition, the product results in an m x q matrix C.

E.g. 1 **A** = 3 × 5 matrix and **B** = 5 × 3 matrix ➔ **AB** = 3 × 3 matrix but **BA** = 5 × 5 matrix.

E.g. 2 **C** = 3 × 5 matrix and **D** = 7 × 3 matrix **CD** = undefined but **DC** = 7 × 5 matrix.

- The entry of matrix C in row i and column j is obtained as follows: sum the products formed by multiplying, in order, each entry (that is, first, second, etc.) in row i of A by the *"corresponding"* entry (that is first, second, etc.) in column j of B.

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}$$

- In fact, it's just pairwise inner products. (dot products)

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_{3\times 1} \Rightarrow \begin{bmatrix} \ \end{bmatrix}_{1\times 1}$$

Example 4.E

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & 1 & -3 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ 2 & 2 \end{bmatrix}_{3\times 2} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

Example 4.D

Given the price and the quantities vector,

$$P = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix}$$

What is the matrix representation for total cost?

The total cost is equal to

Example 4.E Given Ax = c as the form of the matrix representation of system of linear equation, write the below system in the matrix form.

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

$$3x + 2y = 7$$

$$5x - 7y + z = 0$$

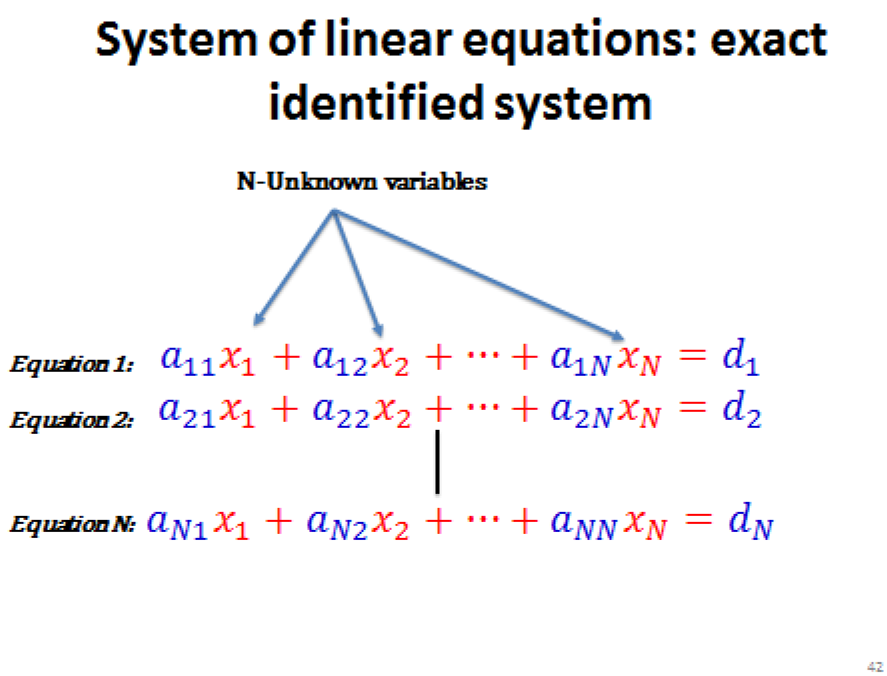
4.5) System of simultaneous equations and Matrix approach to the solution

A system of linear equations typically comprises of **M equations and N unknown.** The N-unknown vector of “x” that satisfy all the M equations simultaneously is called the solution of the system.

Our task is to solve for the N-unknown varaibles!

If exist, matrix procedure allows us to solve for the solution under any arbitrary numbers of M and N. However, we would focus only the case when M =N. That is, we concentrate on the *just-identified* system of equation.

Just-(exactly) identified system



Solution to the N-unknown variables can be derived by two methods.

- An inverse matrix approach
- The Cramer’s rule

4.5.1) Inverse matrix method

The system can be compactly written in the following matrix representation

$$Ax = d$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \\ d_N \end{bmatrix}$$

A = matrix of coefficients that are associated with unknown variables

d = matrix of coefficients that are independent of unknown variables

x = vector of unknown variables

= endogenous variables

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To obtain the solution, pre-multiply A with an inverse of A.

$$A^{-1}Ax = A^{-1}d$$

By the definition of inverse matrix A, it must be the case that

$$A^{-1}A = I_N$$

As a result, we yield that

$$x = A^{-1}d$$

Finding the inverse of A:

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

where adjoint of A, i.e. adj(A), is an n x n matrix transpose of the matrix whose entries are each corresponding cofactor.

What is the cofactor matrix?

We need to understand before what *minor* is.

Minor

M_{ij} is the subdeterminant from deleting the i-th row and j-th column.

Example 4.F

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$M_{13} =$

$M_{22} =$

Cofactor matrix is the matrix whose correspondingly value for the i-th row and j-th column is obtained by,

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Example 4.G: Find the cofactor and the adjoint of

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix}$$

Determinant

For N = 2, determinant of any square matrix A can be calculated by:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

Example 4.H

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$$

Definition: A is said to a *singular* matrix if det(A) = 0

Theorem:

If A is a singular matrix, the just-identified system contains infinite number of solution.

Lapace expansion

To find the determinant of any square matrix A of order N (N>2), select any row (or column) of A and *multiply each entry in the row (column) by its cofactor*. The *sum of these products* is defined to be the determinant of A and is called a *determinant of order n.*

Tip!: Pick any row or column whose entry contains the most of zero.

Example 4.I: Find the determinant and inverse matrix

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix}$$

4.5.2) Cramer's rule

The linear system $\boldsymbol{Ax} = \boldsymbol{d}$ has a solution given by *Cramer's formulas:*

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1,2,3,\dots,N$$

where (i) $\boldsymbol{x_i}$ are the unknowns of the system or the i-th entries of x, and

(ii) the matrix A_i is obtained from $\boldsymbol{A}$ by replacing the ***i-th*** column by the vector ***d.***

Example 4.J: Find the solution using the Cramer's rule

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

4.3 Application examples

Note: Mapping the linear economic model in a matrix form

Example 4.K: Using the form Ax = d, write the following market model in the matrix form.

$$Qd = Qs$$

$$Qd = a - bP + cY$$

$$Qs = e + fP + gW$$

when a, b, c, e, f, g are parameters. Y = income and W = weather condition.

Example 4.L

Consider a simple macroeconomic model.

$C = a + bY_d; \quad 0 < b < 1$

$I = I_a + iY; \quad 0 < i < 1$

$G = G_0$

$T = T_0 + tY; \quad 0 < t < 1$

$R = R_0$

$Y_d = Y - T + R$

where R is the government transfer and G is the government purchase. All the remainings are defined as usual.

- State all the endogenous variables in the model

- Is the above system solvable?

- Simplify the system to a 3-variable system that includes Y, C and I.

- Rewrite the system of equations in the matrix form and solve for all the three variables using the Cramer's rule.

Chapter 5: Single-variable optimization: theory and applications

Optimization theory is a mathematical tool developed to solve for the solution of the problem that can be characterized in the following form

$$\max\ (mim)_{x}\ \ f(x)\ ,\ x \in D.$$

- Optimization theory has been extensively applied into economics analysis.
    - We've seen before that equilibrium economic models include several behavioral equations that capture behavior of groups/persons engaged in the model.
    - All these behavioral equations are actually derived from *the first principle* that agents are rational and choose for *optimal decision*, resulting in the behavior that can be captured by the proposed function.
        - Market demand model
            - Demand equation is the result of maximizing utility, subject to budget set.
            - Supply equation is the result of profit-maximizing decision of firm.
        - National income model
            - Aggregate consumption function is derived from the aggregation of individual consumption function which is the result from intertemporal consumption problem.

5.1 Basic calculus: derivative

What is derivative?

- Derivative reflects the responsiveness of a variable attributed to a variable.

For example, consider a case y=f(x).

- Derivative of y with respect to x is denoted by $\frac{dy}{dx}$ $or\ f'(x)$.
- Very much like to the concept of delta change, derivative considers the case for a small change.

By definition, mathematical notion for derivative is limiting case for

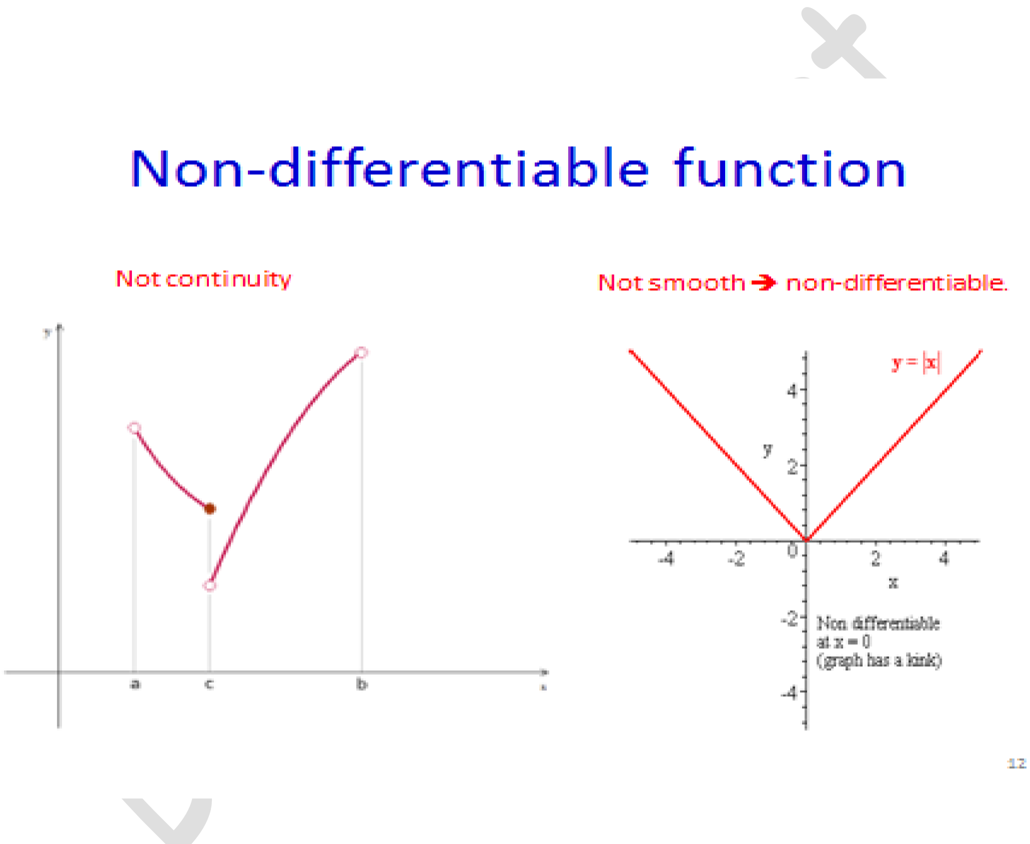
$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x}; (\Delta x \approx 0)$$

Formally stated, the notion of derivative is given by,

$$\begin{aligned}\frac{dy}{dx} = f'(x_0) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\end{aligned}$$

Differentiable function

- For some points of the domain set that defines the function, derivative may not exist.
- This can occurs for two possible reasons.
  - Function is Not continuity (obvious)
  - Function is Not smooth

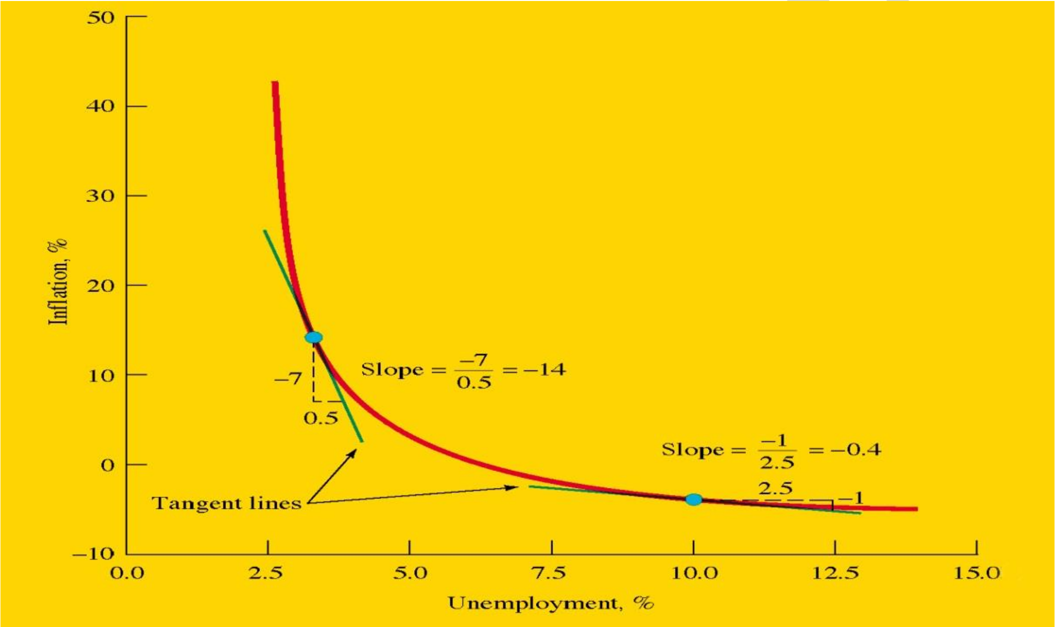


Mathematically, we call a function as a differentiable function if derivative exists for all the domain sets defining that function.

What does the derivative tell us about?

- Since derivative measures the responsiveness when change is very small, its basic idea is related to the concept of "*instantaneous* rate of change", "slope" or more technically termed "tangent line"
- Interpretably, its value tells us both "*magnitude*" and "*direction*" of change in y that is attributed to change in certain direction of x.
  - Increasing vs. decreasing function

Example 5.A: The sacrifice ratio



- Derivative is too used as an input for the approximation problem.
  - To be more precise, this is known as the differential approximation problem.

- Derivative is also helpful for giving some information about the nature of the extreme point.
  - First-order necessary condition suggests that local optimizer is mostly the stationary point of the function, i.e. the point whose tangent line is paralleled to the horizontal axis.

5.2) Rule of differentiation

RULE 1 Derivative of a Constant:

$$\frac{dc}{dx} = 0$$

RULE 2 Derivative of x^n:

$$\frac{dx^n}{dx} = nx^{n-1}$$

RULE 3 Exponential

$$\frac{d(a^x)}{dx} = a^x \ln a$$

RULE 4 Logarithm

$$\frac{d(\log_a x)}{dx} = \frac{1}{x} * \frac{1}{\ln a}$$

RULE 5 Sum or Difference Rule

$$\frac{\mathrm{d}[\mathrm{f}(\mathrm{x}) \pm \mathrm{g}(\mathrm{x})]}{\mathrm{dx}} = \mathrm{f}'(\mathrm{x}) \pm \mathrm{g}'(\mathrm{x})$$

RULE 6 Constant Factor Rule:

$$\frac{d[cf(x)]}{dx} = cf'(x)$$

Rule 7 Product rule

$$\frac{d(f * g)}{dx} = f'(x)g(x) + g'(x)f(x)$$

Rule 8 Quotient rule

$$\frac{d\left(\frac{f}{g}\right)}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Rule 9 Chain rule

$$\frac{df\big(g(x)\big)}{dx} = f'\big(g(x)\big) * g'(x)$$

Derivative: higher order

- Higher order derivative.
  - 2^{nd} order ➔ $f''(x) = \frac{df'(x)}{dx} = \frac{d^2f(x)}{dx^2}$
  - 3^{rd} order ➔ $f^3(x) = \frac{df''(x)}{dx} = \frac{d^3f(x)}{dx^3}$
  - n^{th} order ➔ $f^n(x) = \frac{df^{n-1}(x)}{dx} = \frac{d^nf(x)}{dx^n}$

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Example 5.B Make sure you are familiar with all these!

a. $F(x) = 3x^5 + \sqrt{x}$

b. $f(z) = \frac{z^4}{4} - \frac{5}{z^{1/3}}$

c. $y = 6x^3 - 2x^2 + 7x - 8$

$$F(x)=(x^2+3x)(4x+5)$$

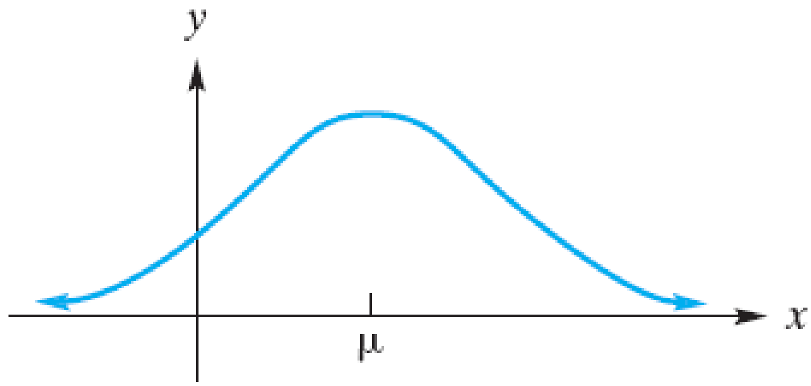
$$F(x)=\frac{4x^2+3}{2x-1}$$

$$y=(x^2-4)^5(3x+5)^4$$

Example 5.C

Given a normal distribution function

$$y = f(x) = \frac{1}{\sigma\sqrt{2x}} e^{-\left(\frac{1}{2}\right)((x-\mu)}$$



Determine the rate of change of y with respect to x when $x = \mu + \sigma$.

Example 5.D

Find an equation of the tangent line to the curve $y = \dfrac{3x^2 - 2}{x}$ *when x = 1.*

Derivative for some economics functions

C(y) = consumption function

- Marginal propensity to consume = C'(y)

C(q) = cost function

- Average cost function: AC(q) = C(q)/q
- Marginal cost function: MC(q) = C'(q)

R(q) = revenue function

- By the construction, R(q) = p*q
- Therefore, average revenue is then R(q)/q. This is equal to "p".
- Marginal revenue is R'(q).

Elasticity:

$$\frac{\Delta y}{\Delta x} * \frac{x}{y} \rightarrow \frac{dy}{dx} * \frac{x}{y}$$

Example 5.D:

Suppose that consumption function takes the form: $C = 10 + 0.1Y + 6\sqrt{Y}$

Find the value of MPC when Y = 100 and 1000.

Example 5.D:

If a manufacturer's average-cost equation is

$$\bar{c} = 0.0001q^2 - 0.02q + 5 + \frac{5000}{q}$$

Find the marginal-cost function. What is the marginal cost when 50 *units are produced?*

Example 5.E

Given the demand function, $p = \frac{k}{q}$ where $k > 0$ and $q > 0$

Show that the demand has constant unit elasticity

5.3) Differential and approximation

Differential

Derivative/differentiation:

$$\frac{dy}{dx} = f'(x_0)$$

Differential in y is given by:

$$dy = f'(x_0)dx$$

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Differential is the linear approximation around x_0. Why? And How?

Example 5.F: Approximate the value of y for each of these values of x.

$$y = \ln(1+x)$$

$x = 0$ ➔

$x = 1$ ➔

$x = 0.01$ ➔

Accuracy of the approximation

- Accuracy of the approximation increases as Δx approaches zero.
- When Δx gets bigger in the size, accuracy of the linear approximation gets worse.
- To fix the problem, one needs to rely on the **higher order** approximation: the Taylor's approximation.

Differential: *Taylor's approximation*

- $f(x) \approx$
$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 + \frac{1}{4!}f^4(x_0)(x - x_0)^4 + \cdots + \frac{1}{n!}f^n(x_0)(x - x_0)^n$$

! = Factorial

N-th ordered polynomial term

N-th ordered derivative

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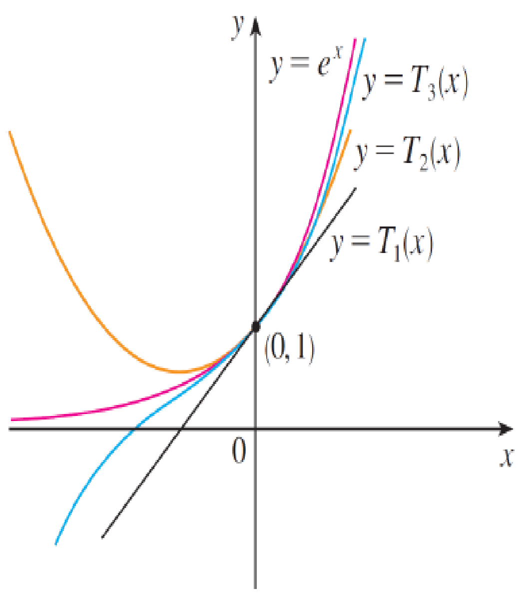
Example: $y = e^x$, **approximate around x = 0**



	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

5.3 Extreme-point problem

5.3.1) Statement of the problem

Suppose $f(x)$ is an objective function.

Mathematical formulation of the extreme point problem is given by

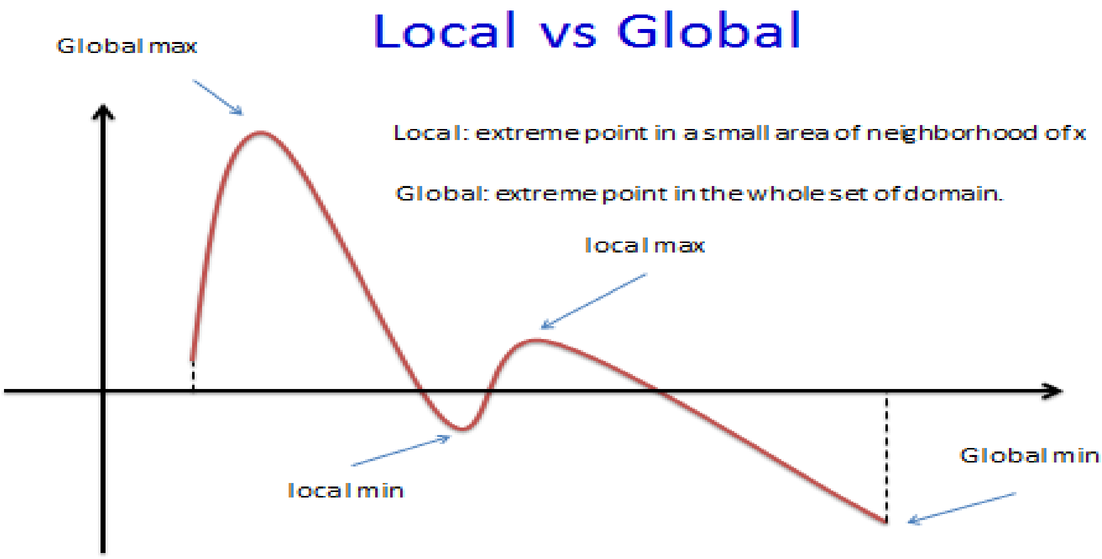
$$\max\ (min)_x \quad f(x) \quad ; \qquad x \in D$$

5.3.2) Types of extreme point

- Local extreme point (optimizer)
  - For all the "X" locating in the neighborhood of x*, the x* is a local optimizer if

    Neighborhood of x =* $\{x / |x - x^*| < \varepsilon\}$

    - Local max: $f(x^*) > f(x)$.
    - Local min: $f(x^*) < f(x)$.
- Global extreme point (optimizer)
  - For all the "X" in the domain defining "f" ($D = D_f$), the x* is the global optimizer if
    - Global max: $f(x^*) > f(x)$
    - Global min: $f(x^*) < f(x)$



Note:

- In general, optimizer may not exist. (When can the existence be warranted: continuous function with compact domain sets.)

- Suppose it does exist, how to locate the points?
  - Calculus provides some important toolkits for searching the local optimizer.
  - In economics, we try to locate is the global optimizer!! So, the problem itself is very hard because the toolkit is only applicable for detecting local optimizer.

- However, under some regularity conditions on the given function and the domain set in consideration, local optimizer can be warranted to be the global optimizer. (What are the conditions?)

- To understand the tool that we need for solving the optimization problem, we need to develop some basic understanding on the characteristics of function implied by the derivative.

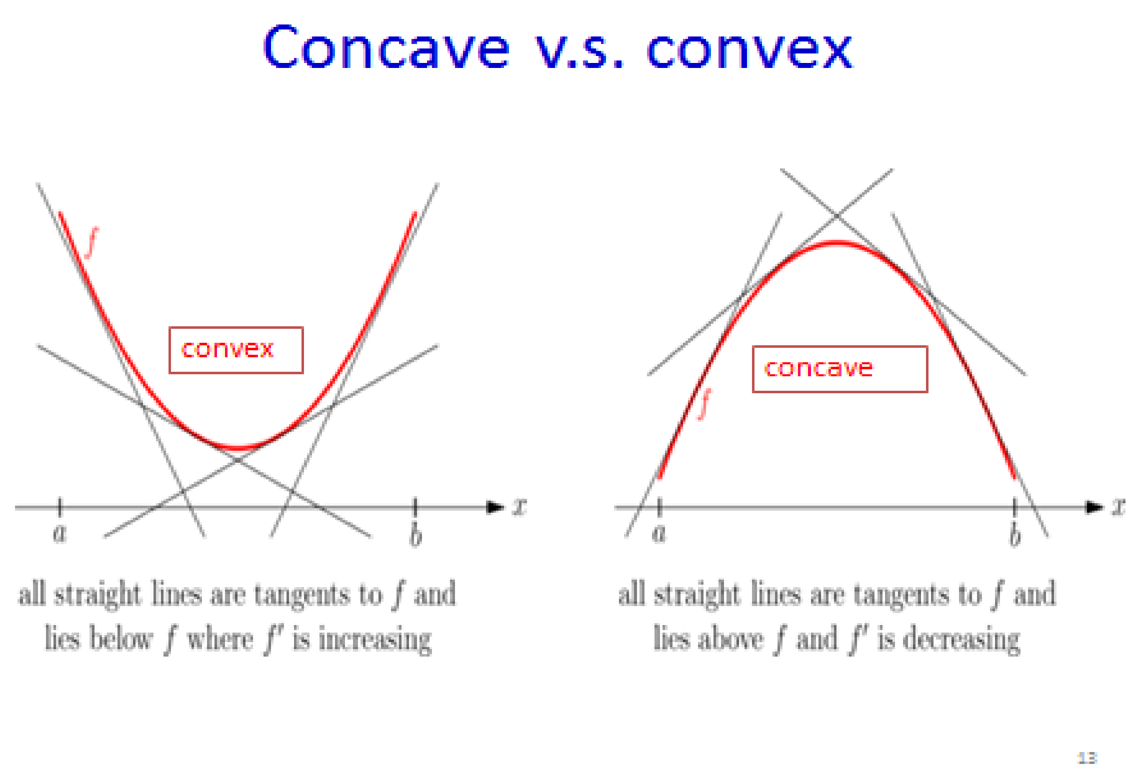
5.3.2) Characteristic of functions by derivative

5.3.2a) Increasing vs decreasing

- Definition: A function is said to be increasing (decreasing) if and only if for any $x_1 > x_2$, we have that $f(x_1) \geq (\leq) f(x_2)$, and vice versa.
  - Strictly increasing (decreasing) = the condition holds with strict inequality.
- Equivalent results for increasing (decreasing) function:
  - Slope is positive (negative)
  - $f'(x) \geq (\leq) 0$ (with strict inequality for "strictly" increasing/decreasing function)

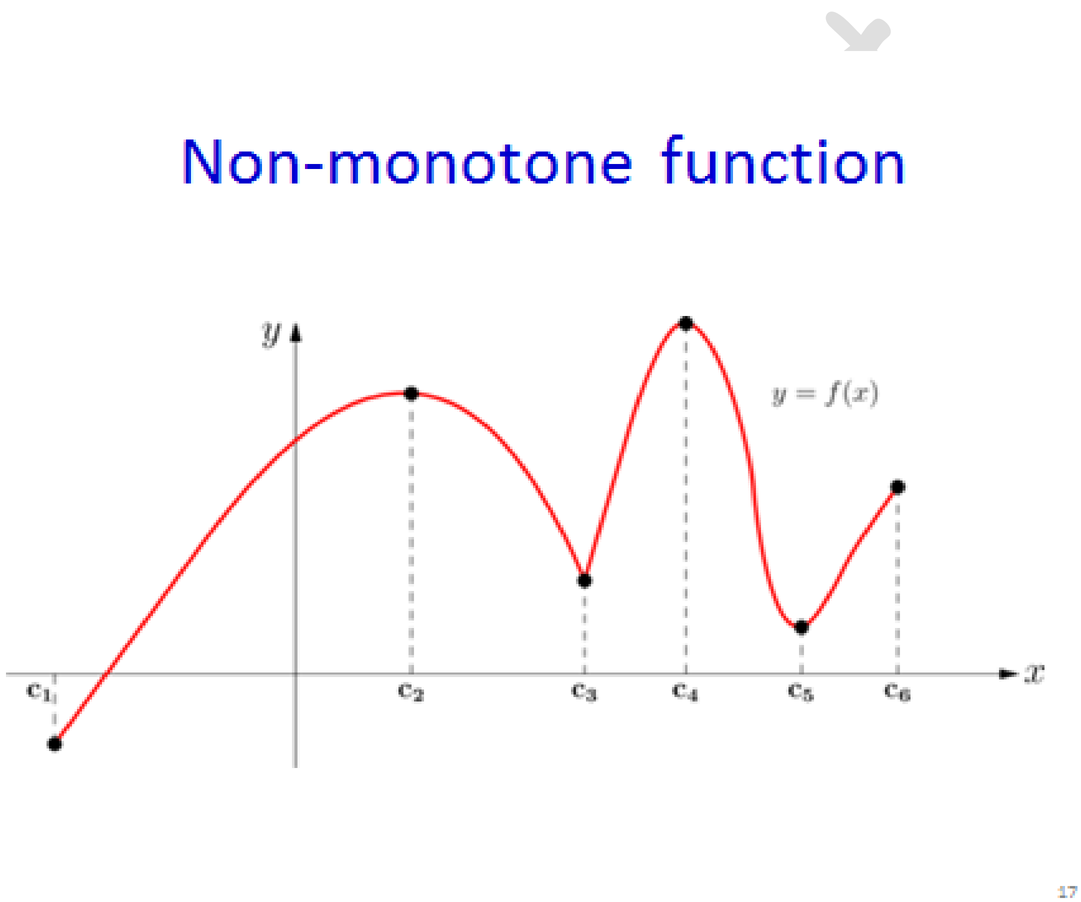
5.3.2b) **Concave and convex function**

- A function is said to be concave (convex) if
  - Slope of f(x) decreases (increases) as X increases.
  - Tangent line is above (below) the curve.
  - $f''(x) \leq 0$.
- If to be a “strict” concave (convex) function, the last condition is modified to hold for only “strict” inequality, i.e. $f''(x) < (>)\ 0$



5.3.1c) Monotone function

- Monotone function = one single property for the function
- E.g. Monotonically increasing function ➔ only increasing function for the whole domain set.
- E.g. Monotonically convex function ➔ convex function for the entire domain set.



5.3.2) Local optimizers

- Calculus only provides tool set for solving *local extreme point*.

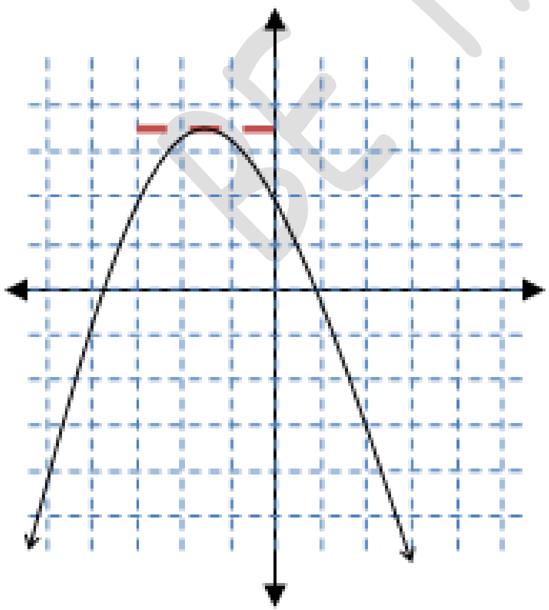
- If one wants to solve for global extreme points, one needs to check all possible local extreme points, and boundary values of the domain set that defines the function.

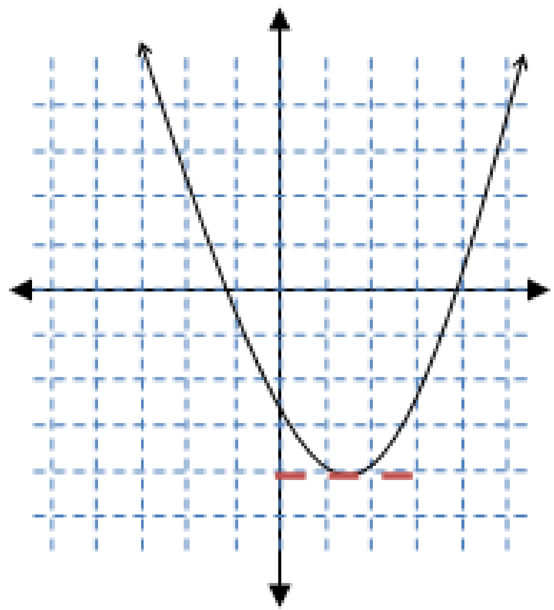
- Under some regularities conditions, *local extreme point is warranted to be global extreme point.*

  Mathematician notices some properties of the local optimizers for the twice-differentiable function. Therefore, they construct an important theorem that characterizes the property of location optimizers. This is known under first-order and second-order necessary condition.

Theorem: *First order necessary condition*

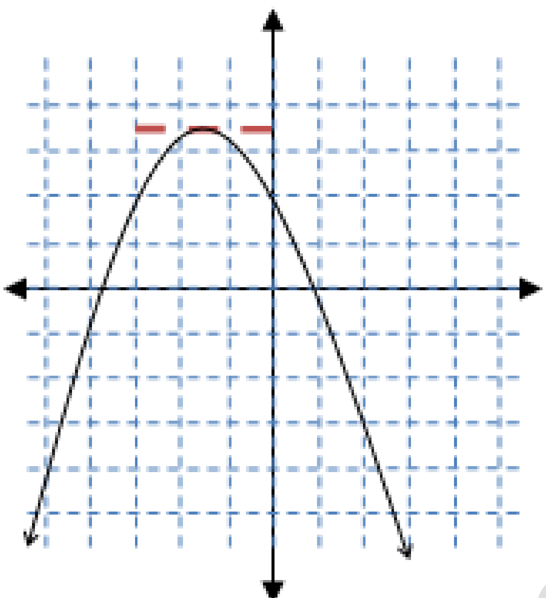
Suppose that f is differentiable function. If x^* is local optimizer, then $f'(x^*) = 0$. That is, x^* is also a stationary point

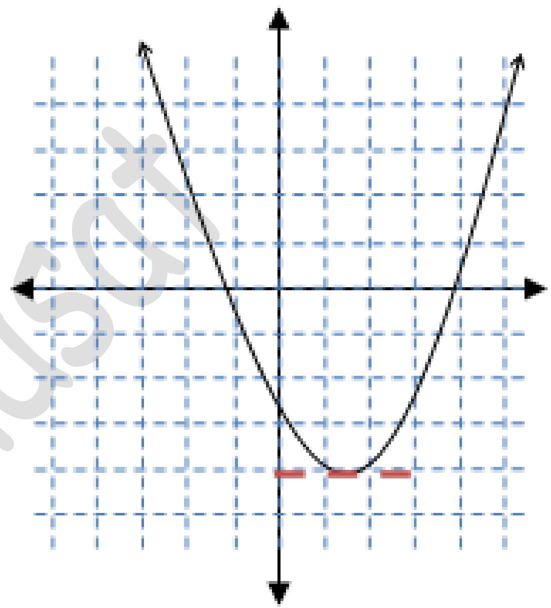




Theorem: *Second-order necessary condition*

Suppose that f is twice-differentiable function. If x^* is local maximizer (minimizer), then $f''(x^*) < (>)\ 0$. That is, x^* is located under the concave (convex) region of the function.





Proof:

By the Taylor's approximation, we know that

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \tfrac{1}{2} f''(x^*)(x - x^*)^2$$

- The approximation should be very accurate because we are considering the value of x in the neighborhood of x*

From the first-order necessary condition, we know that local optimizer is a stationary point, and hence $f'(x^*) = 0$. Therefore, the approximate form can be further simplified into:

$$f(x) - f(x^*) = \frac{1}{2} f''(x^*)(x - x^*)^2$$

Suppose that x^* is a local minimizer, the above equation would imply that

$$\frac{1}{2} f''(x^*)(x - x^*)^2 > 0$$

Since $(x - x^*)^2$ is always positive, the term $f''(x^*)$ must be positive; this is to ensure that above inequality holds. Consequently, the local minimizer, x^*, must be located under the convex region of the function

Cooked procedure:

Cooked procedure: Max versus Min

- If "f" is twice-differentiable function, i.e. $f''(*)$ exists for all the values of x,
  - First-order condition: $f'(x^*) = 0$
  - Second-order condition
    - Maximizer: $f''(x^*) < 0$ (decreasing slope)
    - Minimizer: $f''(x^*) > 0$ (increasing slope)

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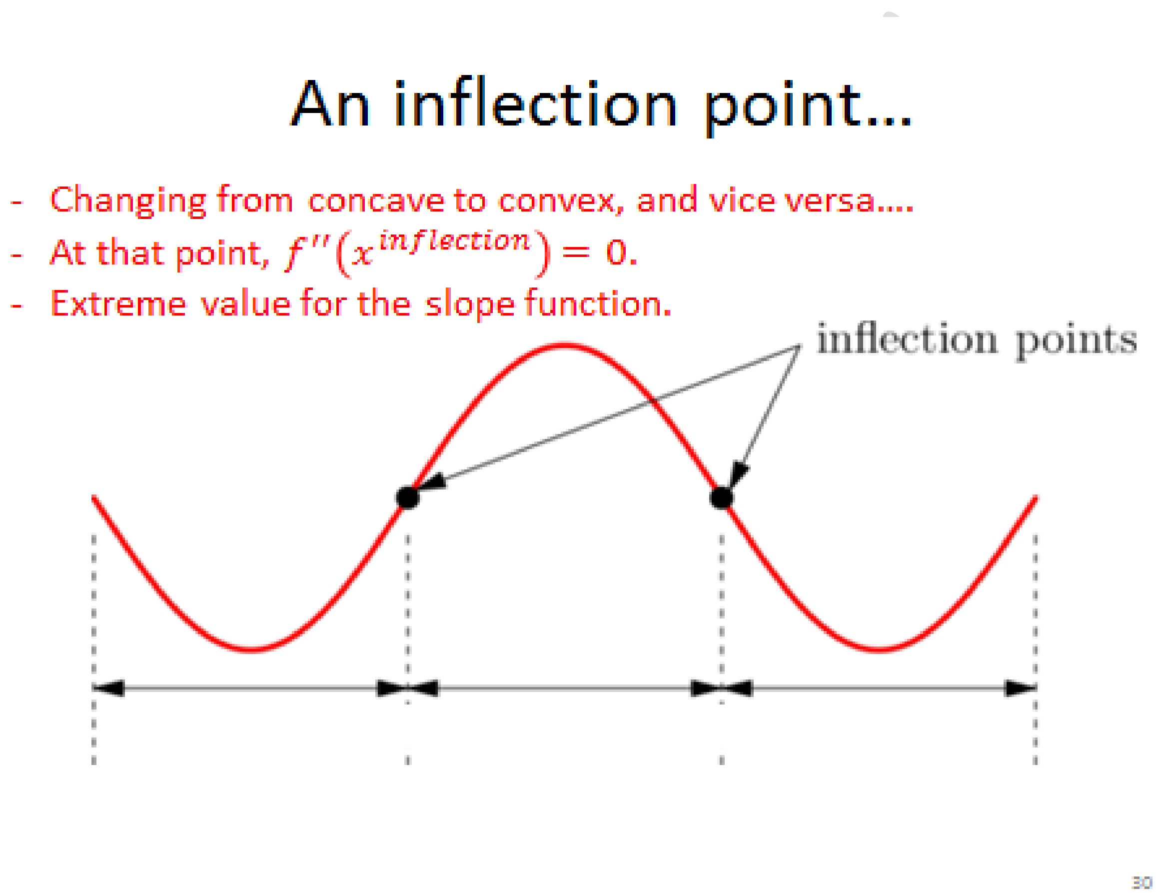
Example 5.G Locate all the local optimizers of $y = x^3 - 3x$. Also, state the type of local optimizer that you calculate.

Inflection point

$x^{inflection}$ is said to be an inflection point when the curvature of the function changes from convex (concave) to concave (convex) at $x = x^{inflection}$.

- $f''\left(x^{inflection}\right) = 0$.

- Extreme value for the slope function



Example 5.H: Given $y = x^4 - 2x^2 + 1$, find local optimizer and inflection point

N-th derivative test

- In some case, it's possible that x^*

  $f'(x^*) = 0$

  $f''(x^*) = 0$

  $x^* =?\, optimizer \;\; or \;\; inflection\, point$

Consider for example $y = x^5 + 1$

$f'(0) = 0$

- X = 0 is a critical (stationary) value. So, one could think that this is a candidate for local optimizer.

$f''(0) = 0$

- But, this is also a possible candidate for an inflection..

Then, what is it actually?

The n-th order derivative test..

- For the x^* with $f'(x^*) = 0$, and other higher-order derivatives are all also zero, but the N-th
  - If N is an even number,
    - » x^* is local max if $f^N(x^*) \neq 0 < 0$.
    - » x^* is local min if $f^N(x^*) \neq 0 > 0$.
  - If N is odd number, x^* is an inflection point.

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Example 5.I: $y = x^5 + 1$ v.s. $y = x^4 + 1$

Global optimizer

- Normally, it's very hard to locate where the point is.
- Most numerical work usually ends up wrongfully yielding the local optimizer as the answer; this is wrong.
- Engineering heavily relies on this technique, and extensively develop tools for identifying/detecting the global optimizers under more robust conditions.

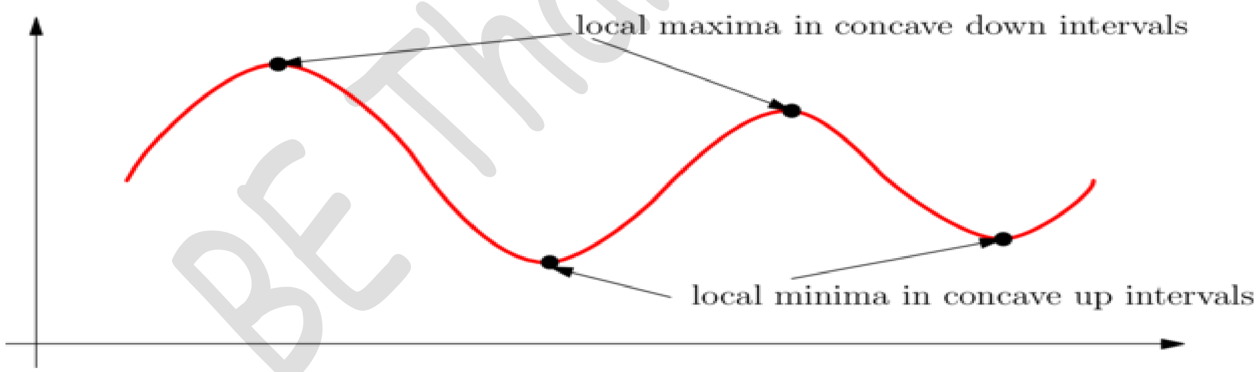
Weistrass theorem

Under a closed and bounded domain set of a continuous function, global optimizer is guaranteed to exist!

- It may *or* may not exist for the opened domain!

To find the global optimizers:

- Calculate the value of ***all the local optimizers*** and ***the two-value at the boundary.***



Example 5.J: $y = x^3 - 3x; \quad x \in [-2, 2]$

Sufficient condition for the local optimizers to be a global one

Assume some nice properties of the function to ensure that local optimizer is automatically global optimizer.

- If function is *monotonically* concave, local maximizer is then global maximizer.
- If function is *monotonically* convex, local minimizer is then global minimizer.

Example 5.K:

- $\mathrm{f}(x) = 2x^2 - 2x + 1$
- $x^* = \frac{1}{2}$ ➔ $\mathrm{f}''(x) = 4 > 0.$
- Monotonically convex➔ global min.

Example 5.L:

- $\mathrm{f}(x) = -e^x + x$
- $\mathrm{f}'(\mathrm{x}) = -e^x + 1 \Rightarrow x^* = 0$
- $f''(x) = -e^x < 0$
- Monotonically concave ➔ global max.

5.4 Some economics applications

- Here are the lists of some of the applications that we will go over in this lecture note.
  - Students are advised to study more by practicing with the problem sets given.

- Competitive equilibrium
  - Firm's optimization.
  - Property of cost minimization
  - Profit-maximizing output.
  - Individual supply curve.
  - Market supply curve.
  - Short-run competitive equilibrium.

- Monopoly equilibrium
  - Profit-maximizing output

5.4.1) Competitive equilibrium

- A model that predicts the equilibrium of market when each individual takes price as given.
    - Welfare theorist proves that this market structure results in the *first-best allocation*.

- In previous part, we've seen before the technique to solve for market equilibrium.
    - This requires the information about market demand curve and market supply curve.

- We knew that both market demand and market supply are simply by-product of the aggregation of each individual demand and individual supply.

- Individual supply can be derived from the profit maximization problem!

Profit maximizing problem:

Firm's problem is to choose for Q that maximizes profit.

$$\pi = R(Q) - C(Q)$$

$R(Q)$ differs across market structure.

- Can firm set price? Is the firm taking price as given?

- Competitive firms: $R(Q) = PQ$

- Monopoly: $R(Q) = P(Q) * Q$

We will now consider the case for competitive firms. Our goal is to answer the following question: *how to pin down the optimal level of output that maximizes the profit function.*

Optimality conditions for PMP

First-order optimality condition:

$\pi'(Q^*) = 0 \quad \Rightarrow \quad R'(Q) - C'(Q) = 0$

$\Rightarrow \quad MR(Q^*) = MC(Q^*)$; setting Q where MR = MC

$\Rightarrow \quad P = MC(Q^*)$

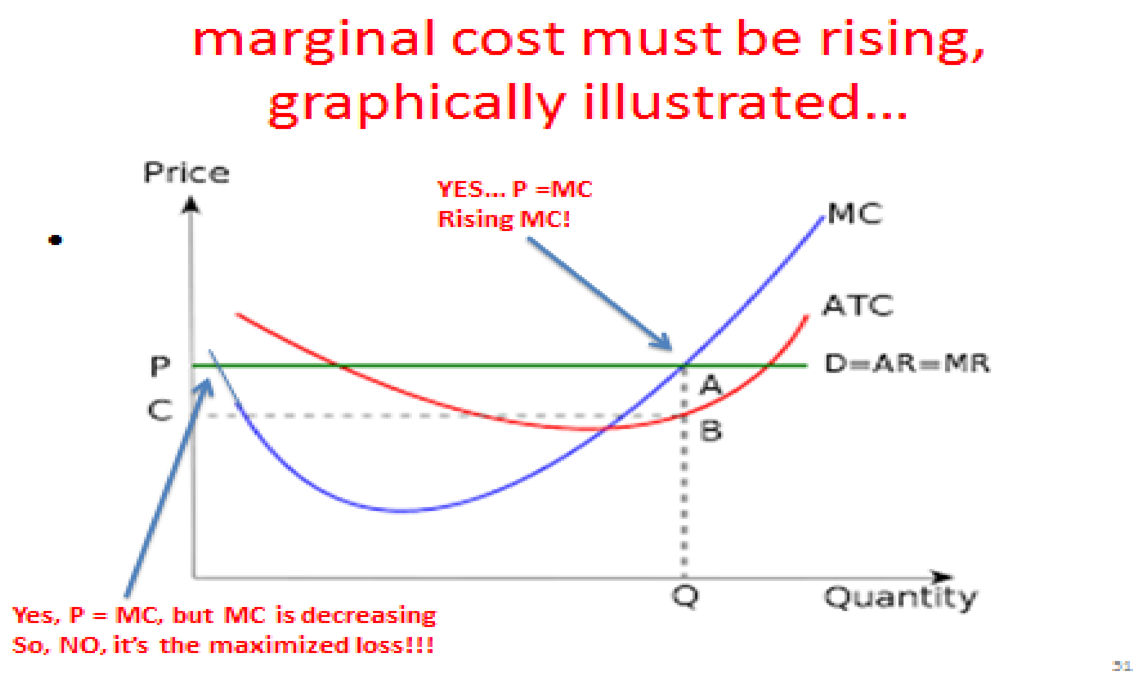
Second order condition

$\pi''(Q^*) < 0$; profit function is convex!

$\Rightarrow \quad MR'(Q^*) - MC'(Q^*) < 0$

$\Rightarrow \quad -MC'(Q^*) < 0$

$\Rightarrow \quad MC'(Q^*) > 0$; Setting Q where marginal cost is rising

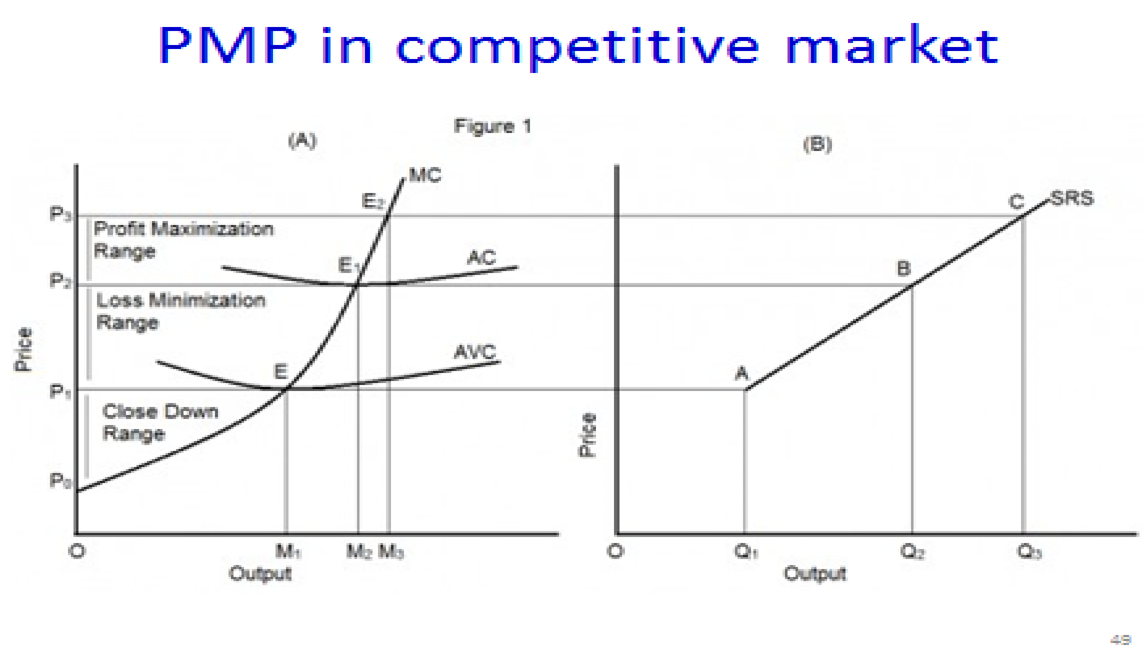


Example 5.M: Suppose that cost function takes the following equation: $AC(Q) = 0.2Q + 4 + \frac{400}{Q}$

- What is the level of profit-maximizing output when P = 8?

The short-run supply curve

- The *short-run* supply curve is the quantity-price schedule that results from the profit maximization problem.
    - Conceptually, we yield different level of optimal outputs for different prices.
    - Since it requires that we choose to the level of output where price is equated to marginal cost under the rising part of marginal cost, optimal quantity-price schedule (the supply relationship) must be increasing.



Shut-down

- Supply curve is not the entire curve of the marginal cost curve.
    - If price is dropping too low, firm might choose to temporarily stop its production.
    - This is called the *shut-down price*.
    - Shut-down price occurs when P = min(AVC) at that Q.
- Not to be confused with the break-even point.
    - Break-even point occurs when P = min(AC)

Example 5.M (contd): Use the information above and derive the supply equation. Properly characterize the region under which firm is active in the market.

Definition: Given N firms in the market and market demand, short-run competitive equilibrium comprises of allocation "Q^e", $\{q_1^e, q_2^e, \ldots, q_N^e\}$ and "P^e" such that

(i) q_i^e is the profit-maximizing level of output at P^e.

(ii) $Q^e = q_1^e + q_2^e + \cdots + q_N^e$

(iii) P^e clears the market, i.e. $Q^e = demand$ at the equilibrium price

Equilibrium notion is somehow confusing. However, solving for the competitive equilibrium is very easy.

- Derive individual supply
- Aggregate up to derive market supply
- Solve for price that clears that market, i.e. supply = demand

---

Example 5.N: Suppose that $TC(q) = 0.1q^2 + 10$ Consider the following problem

- Derive individual supply equation.

- Suppose that there are 5 identical firms. Derive market supply equation.

Given that there are two identical consumers in the market with individual demand equation given by $P = 10 - q$.

Solve for the market equilibrium.

How much is the profit that each firm earns in the equilibrium?

Long-run equilibrium

- In the long-run, firms can freely enter or exit the industry.
    - If profit arises, outsiders will be attracted into the market.
    - If loss incurs, insiders will leave the industry

- Given that firms are driven by profit opportunity, the number of firm will be an endogenous variable.

- Long-run equilibrium would occur when each existing firm in the market receives zero profit. (no incentive to enter/exit!)

Definition:

Given the market demand, long-run competitive equilibrium comprises of allocation "Q^e", $\{q_1^e, q_2^e, \ldots, q_{N^e}^e\}$, "$N^e$", and "$P^e$" such that

(i) q_i^e is the profit-maximizing level of output at P^e.

(ii) $Q^e = q_1^e + q_2^e + \cdots + q_{N^e}^e$

(iii) P^e clears the market, i.e. $Q^e = demand$ at the equilibrium price

(iv) N^e results in zero-profit condition.

Algorithm:

- Each firm earns zero profit when each of them produce at the break-even price, i.e P = min(AC). At the break-even price, one also knows the break-even output for each firm.
- As a result, market equilibrium must be the break-even price.
- Given the break-even price, one can figure out total demand.

5.4.2) Monopoly

- Single firm in the market; barrier to entry
- Take control the whole market; with the ability to set price along the market demand curve.

With the monopoly,

$R(Q) = P(Q) * Q$

Optimality condition implies the level of output such that

MR(Q) = MC(Q)

Example 5.p: Given market demand is $p = 10 - Q$ and

$$TC = \frac{1}{3}Q^3 - Q^2 + 6Q$$

Find the level of profit-maximizing output, and price that producer would charge.

What do we know about the pricing policy of the monopoly?

- We can show that optimal behavior of monopoly can be alternatively characterized as the optimal mark-up (P/MC).

Example 5.o: Show that optimal behavior of the monopolist would entail them to choose for optimal mark-up where the level of mark-up is inversely determined by elasticity of demand.