Chapter 3

Discrete Random Variable and Their Probability Distributions

- Basic Definition
- The Probability Distribution for a Discrete Random Variable

Basic Definition

A random variable (rv) is a real-valued function defined over a sample space. Consequently, a random variable can be used to identify numerical events that are of interest in an experiment.

Such a numerical value will depend on the outcome, which in turn are governed by an underlying probability model (or distribution).

Definition 1 : A random variable Y is said to be discrete if it can assume only a finite or countably infinite number of distinct values.

For example, the event of interest in an opinion poll regarding voter preferences. Let Y = the number of voters favoring a certain candidate or issue. The observed value of this random variable must be 0 or an integer between 1 and the sample size. Thus, this random variable can take only a finite number and is said to be discrete.

The Probability Distribution for a Discrete Random Variable

Notations :

- Uppercase letter (*Y*) – random variables
- Lowercase letter (*y*) – particular values that a random variable may assume
- The expression *(Y=y)* can be read, the set of all points in *S* assigned the value y by the random variable Y.
- *P(Y=y)* = the probability that *Y* takes on the value *y*.

Definition 2 : The probability that Y takes on the value y, $P(Y=y)$, is defined as the *sum of the probabilities of all sample points in S* that are assigned the value y. We will sometimes denote $P(Y=y)$ by $p(y)$.

Definition 3 : The *probability distribution* for a discrete variable Y can be represented by a formula, a table, or a graph, which provides $p(y) = P(Y=y)$ for all y.

Examples

<u>Example 1</u> : Draw 2 balls and win the sum. Fee is $2. Interested in the net gain.

- 9 balls of zero
- 2 balls of three
- 1 ball of five

Let Y = net gain (in $) - random variable

fee

Then Y = sum of two numbers drawn – 2

(i) Determine all possible values of Y

(ii) Corresponding probabilities associated with each value of Y.

Note :

The previous table which includes a listing of all distinct values of a random variable (in this case Y) and their corresponding probabilities is called a <u>probability distribution for Y</u>.

Examples

Example 2 : A supervisor of a manufacturing plant has three men and three women working for him. He wants to choose two workers for a special job. Not wishing to show any biases in his selection, he decides to select the two workers at random. Let Y denote the number of women in his selection. Find the probability distribution for Y.

Solution :

- Let Y = number of women in his selection
- The number of ways of selection two workers from six workers is
- The values for Y are
- The number of ways of selecting Y = 0 women is

Theorem 1 : For any discrete probability distribution, the following must be true :

(1) $0 \leq p(y) \leq 1$ for all y.

(2) $\sum_{y} p(y) = 1$, where the summation is over all values of y with nonzero probability.

The Expected Value of a Random Variable or a Function of a Random Variable

In this section, we are interested in the "mean" or average of a discrete random variable which is commonly referred to as "Expected value".

Definition : Let Y be a discrete random variable with the probability function $p(y) = P(Y=y)$. Then the *expected value* of Y, $E(Y)$, is defined to be

$$\mu = E(Y) = \sum_y y p(y)$$

Example

Example 3 : Refer to Example 1. We are interested in determining if the game is fair.

(For this we need to compute the expected net gain . Ie. We want E(Y).)

probability distribution

Y = y	-2	1	3	4	6
P(y)	72/132	36/132	18/132	2/132	4/132

More generally, we are interested in not only the expected value of a given random variable, but also the expected value of a function of a random variable.

Most notable example of this is the variance of a random variable.

Theorem 2 : Let Y be a discrete random variable with probability function $p(y)$ and $g(Y)$ be a real-valued function of Y. Then the expected value of $g(Y)$ is given by

$$E[g(Y)] = \sum_{y} g(y)\,p(y).$$

Definition 5 : The variance of a random variable Y is defined to be the expected value of $(Y-\mu)^2$. That is,

$$V(Y) = E\left((Y-\mu)^2\right).$$

The standard deviation of Y is the positive square root of $V(Y)$.

Example

Example 4 : The probability distribution for a random variable Y is given below. Find the mean, variance, and standard deviation of Y.

y	2	4	6
p(y)	.2	.3	.5

Theorem 3 : Let Y be a discrete random variable with probability function $p(y)$ and c be a constant. Then $E(c) = c$.

Proof

Theorem 4 : Let Y be a discrete random variable with probability function p(y), g(Y) be a function of Y, and let c be a constant. Then

$$E\left[cg(Y)\right]=cE\left[g(Y)\right].$$

Proof :

Theorem 5 : Let Y be a discrete random variable with probability function $p(y)$ and $g_1(Y)$, $g_2(Y)$, …, $g_k(Y)$ be k functions of Y. Then

$$E\left[g_1(Y)+g_2(Y)+\ldots+g_k(Y)\right]=E\left[g_1(Y)\right]+E\left[g_2(Y)\right]+\ldots+E\left[g_k(Y)\right].$$

Proof :

Theorem 6 : Let Y be a discrete random variable with probability function p(y); then

$$\sigma^2 = V(Y) = E\left((Y-\mu)^2\right) = E(Y)^2 - \mu^2 .$$

Proof :

Examples

Example 5 : Refer to Example 1. Draw 2 balls and win the sum. Fee is \$2. Interested in the net gain.

- 9 balls of zero
- 2 balls of three
- 1 ball of five

Let Y = net gain (in \$)

Then Y = sum of two numbers drawn – 2

Calculate V(Y) for this example.

Solution :

The probability distribution is

$Y = y$	-2	1	3	4	6
$P(y)$	72/132	36/132	18/132	2/132	4/132

Example 6 : Refer to Example 4. The probability distribution for a random variable Y is given below. Find variance of Y by using Theorem 3.6 .

y	2	4	6
p(y)	.2	.3	.5

Solution :

Example 7: A box contains 7 red balls, 3 black balls, 2 balls are randomly selected without replacement.

(a) Find the probability distribution for the number of red balls.

(b) Find the mean of the random variable represented by the number of red balls.

(c) Find the standard deviation for the number of red balls.

Solution : Let X = # of red balls

The Binomial Probability Distribution

Review

Recall that a binomial experiment satisfies :

(1) Fixed number (n) of trials.

(2) Each trial has only 2 possible outcomes.
Success (S)
Failure (F)

(3) $P(S) = p$ => same on every trial

$P(F) = 1 - p = q$.

(4) the trials are independent.

(5) The random variable of interest is Y, the number of successes observed during the n trials.

Binomial Distribution

<u>Definition 7</u> : A random variable Y is said to have a *binomial distribution* based on *n* trials with success probability *p* if and only if

$$p(y) = \binom{n}{y} p^y q^{n-y} \quad , \qquad y = 0, 1, 2, \ldots, n \text{ and } 0 \le p \le 1$$

- call Y a binomial random variable

Binomial Expansion

Recall the Binomial expansion Theorem.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}y^n$$

Observe that

$$\sum_{y=0}^{n} p(y) = \sum_{y=0}^{n} \binom{n}{y} p^y q^{n-y} = (p+q)^n = (p+1-p)^n = 1$$

sum to 1

Theorem 7 : Let Y be a binomial random variable based on n trials and success probability p. Then

$$\mu = E(Y) = np \quad \text{and} \quad \sigma^2 = V(Y) = npq$$

- proof is in the text, but is omitted here.

Examples

Example 8 : Experience has shown that 30% of all persons afflicted by a certain illness recover. A drug company has developed a new medication. Ten people with the illness were selected at random and received the medication; nine recovered shortly thereafter. Suppose that the medication was absolutely worthless. What is the probability that at least nine of ten receiving the medication will recover?

Solution :

- Let Y denote the number of people who recover.
- n =
- P(success) = probability that a single ill person will recover =

Want : $P (Y \geq 9) =?$

Example 9 : A multiple-choice examination has 15 questions, each with five possible answers, only one of which is correct. Suppose that one of the students who takes the examination answers each of the questions with an independent random guess. What is the probability that he answers at least ten questions correctly?

Solution :

Example 10 : Vaccinate children for a disease, and the life time immunity is found to be 70%. If 3 children are vaccinated.

(1) Find the probability that the first child immunized but the next two did not.

(2) Find the probability that the only second child immunized.

(3) Find the probability that exactly 3 children immunized.

(4) Find the probability that at least 2 children immunized.

(5) If 10 children are vaccinated. Find the expected number to immunize and the standard deviation of number of children immunized.

Solution : Let

X = total # of children immunized

The Poisson Probability Distribution

Poisson Probability Distribution

<u>Definition</u> : A random variable Y is said to have a Poisson probability distribution if and only if

$$P(Y=y)=p(y)=\frac{\lambda^{y}}{y!}e^{-\lambda} \quad , \quad y=0,1,2,\ldots$$

$$\lambda>0 \qquad e=2.718\ldots$$

Remarks :

(1) Certainly P(Y = y) $\geq$ 0 , since $\lambda^y > 0$, $e^{-\lambda} > 0$

(2) Taylor Series for e^y :

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \qquad \text{for all } y$$

(3) Claim : $\displaystyle\sum_{y=0}^{\infty} P(Y = y) = 1$

$$\sum_{y=0}^{\infty} P(Y = y) = \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

$$= \left(e^{-\lambda}\right)\left(e^{\lambda}\right) = 1$$

The Poisson probability distribution often provides a good model for the probability distribution of the number Y of rare events that occur in space, time, volume, or any other dimension, where λ is the average value of Y. The examples of random variables with approximate Poisson distributions are

(1) the # of telephone calls handled by a switchboard in a time interval,

(2) the #of radioactive particles that decay in a particular time period,

(3) the # of errors a typist makes in typing a page,

(4) the # of automobiles using a freeway access ramp in a 10-minute interval.

etc.

Theorem 11 : If Y is a random variable possessing a Poisson distribution with parameter λ, then

$$\mu = E(Y) = \lambda \text{ and } \sigma^2 = V(Y) = \lambda$$

Proof :

$$E(Y(Y-1)) = \sum_{y=0}^{\infty} y(y-1)\,p(y)$$

$$= \sum_{y=0}^{\infty} y(y-1)\frac{\lambda^y e^{-\lambda}}{y!} = \sum_{y=2}^{\infty} \frac{\lambda^y e^{-\lambda}}{(y-2)!}$$

$$= \lambda^2 \sum_{y=2}^{\infty} \frac{\lambda^{y-2} e^{-\lambda}}{(y-2)!} = \lambda^2(1) = \lambda^2$$

$$E(Y(Y-1)) = E(Y^2 - Y) = E(Y^2) - E(Y) = E(Y^2) - \lambda$$

$$\Rightarrow E(Y^2) = \lambda^2 + \lambda$$

So $V(Y) = E(Y^2) - \left(E(Y)\right)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Important Fact :

If Y = total # of happenings that occur randomly and independently in a time (area, volume) window of size t units and with at the average rate of δ per unit time (area, volume), then Y is a Poisson random variable with parameter $\lambda=\delta t$

Examples

Example 19 : Suppose heart attack emergencies at a local hospital occur at random times at the average rate of 2 per day. Let Y = total # of heart attacks emergencies at the hospital during a certain 5-day period. Find

(a) $P(Y \geq 15)$.

(b) $P(Y \leq 5 \mid Y \leq 9)$

(c) P(exactly one of the 5 days will have no heart attack emergencies)

Example 20 : Suppose a .001 (mm^3) sample of blood from a patient with white cells count 6000 per mm^3 is analyzed. Let Y = total # of white cells in the blood sample. Assuming white cells are randomly distributed in blood, compute :

(a) $E(Y)$, $V(Y)$

(b) $P(5 \le Y \le 7)$

(c) $P(Y = 8)$

(d) $P(Y \le 8 \mid Y \ge 2)$

Example 21 : Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. During a given hour, what are the probabilities that

(a) no more than three customers arrive?

Let Y = # of customers arriving.

(b) at least two customers arrive?

(c) Exactly five customers arrive?

Example 21 : Refer to Example 20. If it takes approximately 10 minutes to serve each customer, find the mean and variance of the total service time for customers arriving during a 1-hour period. (Assume that a sufficient number of servers are available so that no customer must wait for service.)

Let Y = # of customers arriving.

Example 21 (con’t) : Is it likely that the total service time will exceed 2.5 hours?

Example 22 : Refer to Example 21. Find the probability that exactly two customers arrive in the 2-hour period of time :

(a) Between 2.00 P.M. and 4.00 P.M. (one continuous 2-hour period).

Let Y = # of customer that arrive in a given 2-hour period of time.

(b) Between 1.00 P.M. and 2.00 P.M. or between 3.00 P.M. and 4.00 P.M. (two separate 1-hour periods that total 2 hours).

Let Y_1 = # of customers that arrive between 1.00 p.m. and 2.00 p.m.

Y_2 = # of customers that arrive between 3.00 p.m. and 4.00 p.m.