Tax-revenue Maximization.

Suppose $P = a - bQ$ and $TC = c_0 + c_1 Q + c_2 Q^2$.
Suppose a specific tax \$t per unit is imposed.

$\Rightarrow TR = P(Q) \times Q = (a - bQ)Q = aQ - bQ^2$

$\Rightarrow TC_{NO\ tax} = c_0 + c_1 Q + c_2 Q^2$

$\Rightarrow TC_{WITH\ TAX} = c_0 + c_1 Q + c_2 Q^2 + tQ$

$\therefore$ Net profit after tax : $\Pi_N(Q) = (aQ - bQ^2) - (c_0 + c_1 Q + c_2 Q^2 + tQ)$

$$\max_Q \Pi_N = (aQ - bQ^2) - (c_0 + c_1 Q + c_2 Q^2 + tQ)$$

F.O.N.C. : $\frac{d\Pi_N}{dQ} = (a - 2bQ) - (c_1 + 2c_2 Q + t) = 0$

$\Leftrightarrow \frac{d\Pi_N}{dQ} = (-2b - 2c_2)Q + (a - c_1 - t) = 0$

$$\Rightarrow \boxed{Q^* = \frac{a - c_1 - t}{2(b + c_2)}}$$

S.O.S.C : $\frac{d^2\Pi_N}{dQ^2} = -2b - 2c_2 < 0$.

$\therefore$ $Q^* = \frac{a - c_1 - t}{2(b + c_2)}$ is a profit-maximizing quantity.

Given Q^* and tax rate t,
total tax revenue is : $\boxed{T = tQ^* = \frac{(a - c_1 - t)t}{2(b + c_2)}}$

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Maximize total tax revenue:

$$\max_{t} \; T(t) = \frac{(a-c_1-t)t}{2(b+c_2)} = \frac{(a-c_1)t - t^2}{2(b+c_2)}$$

FONC: $$\frac{dT}{dt} = \frac{a-c_1}{\cancel{2(b+c_2)}} - \frac{2t}{\cancel{2(b+c_2)}} = 0$$

$$\Rightarrow \boxed{t^* = \frac{a-c_1}{2}}$$

SOSC : $$\frac{d^2T}{dt^2} = \frac{-2}{2(b+c_2)} < 0.$$

$\therefore$ $t^* = \frac{a-c_1}{2}$ is maximizing total tax revenue.

Example

Given $P = 40 - 0.5Q$, $TC = 2 - 5Q + 7Q^2$. Find t that maximizes T.

$$\max_{Q} \; \pi_N = (40Q - 0.5Q^2) - (2 - 5Q + 7Q^2) - tQ$$

FONC: $$\frac{d\pi_N}{dQ} = 40 - Q + 5 - 14Q - t = 0$$

$$\Rightarrow 15Q = 45 - t$$

$$\boxed{Q^* = \frac{45-t}{15}}$$

$\Rightarrow$ Total tax revenue : $\boxed{T = t \times Q^* = \left(\frac{45-t}{15}\right)t = \frac{45t - t^2}{15}}$

$$\max_{t} \; T(t) = \frac{45t - t^2}{15}$$

FONC : $$\frac{dT}{dt} = \frac{45}{\cancel{15}} - \frac{2t}{\cancel{15}} = 0$$

$$\therefore \boxed{t^* = \frac{45}{2} = 22.5.}$$

SOSC : $\frac{d^2T}{dt^2} = -2 < 0$ ✓.

$\therefore$ $t^* = 22.5$ is the T-max tax rate.