System of equations and Equilibrium model in economics

Topics

- System of equations in mathematics
- Solution method for system of equations
- Mathematical model and system of equations
- Applications
  - Micro: Market equilibrium model
  - Macro: National income model.

System of equations

- Comprises of N-unknown variables **and** M-equations.
  - No need for N to be equal to M.
  - But we need at least N independent equations
  - Equations could be linear, nonlinear, or mixed.
- Set of N-unknown variables that simultaneously satisfies all the M equations, at the same time, is called the solution of the system.

System of equations

- Most equilibrium model studies inter-relationship of economic variables.
- These variables are, most of the time, simultaneously determining each other/one another.
- So, the concept of equilibrium model in economics matches perfectly with the concept of mathematical system that has multiple equations.

System of equations: solution

- For now, we will only focus on
  - 2 x 2 case
  - 3 x 3 case (you will work thorough this case in your homework.)
- These two cases are solvable by pencil and pen.
  - I’m not showing you 3 x 3 because it’s time consuming. Procedurally, it’s the same as 2 x 2 case.
- For any larger scale of system of equations, we have to apply matrix algebra to solve for the solution. Practically, we solve those big-scale system of equations using computer.

System of equations: solution

- At least, make sure you know how to solve for the solution of 2 x 2 case.
- Example:

$$X + Y = 7$$

$$3X - 6y = -15;$$

  - Substitution method (works for 2 x 2 case)
  - Gaussian iterative elimination. (algorithm that computer uses in solving solution for system of linear equations.)

Solving for the solution of system of equations

- Types of solution
- **Example:**
  - System 1: $2x + 3y = 7$ and $x - y = 5$.
  - System 2: $y = -2x + 6$ and $4x + 2y = 12$.
  - System 3: $x + 2y = 14$ and $3x + 6y = 8$
- System 1 has a **unique** solution.
- System 2 has an **infinite** solution
- System 3 has **no** solution.

Application

- Mathematical economic model is usually written in term of system of simultaneous equations in which several variables are simultaneously determine each other, through behavioral equations and equilibrium conditions.

Examples

- Market (partial) equilibrium model
  - Demand equation
  - Supply equation
  - Equilibrium condition
- Multimarket (general) equilibrium model
  - Demand equation for each market
  - Supply equation for each market
  - Equilibrium condition for each market.
- National income model
  - Consumption function
  - National income identity
- The IS-LM model

What we do..

- We will *play* with these models.
- Let’s start from (single) market equilibrium model.
- Our focus would be on the following issues.
  - Determining the market equilibrium
  - Predicting the effect of some *exogenous* changes.
  - Introducing taxation in the model
  - Discuss about tax burdens
- All these in math approach!...

Example: Market equilibrium model

- Suppose that ***market*** demand and supply can be given by the following two equations:

  Qd = 10 - 3P and P = 2 + Qs.

  a. Graph demand and supply curve.
  b. State the equilibrium condition. Find the market equilibrium for quantity of output and price.
  c. Suppose now we generalize that Qd = 30 - 5*income - 3P. At what level of income, does this generalized form coincide with the form of demand equation given above?
  d. Solve for the market equilibrium by using the specification of demand given in “c”. What happen to equilibrium output/price when income is $5 higher than the level associated to that in “c”. Is the product in this question normal/inferior goods?

Example: Market equilibrium model

- Whenever you're asked to draw demand/supply curve, it's by convention that we put "p" on the vertical axis and "q" on the horizontal axis.
- To make your life easier, always rewrite any given form of demand/supply equations into the p-equal form.

Example: Market equilibrium mode

- Demand: $P = (10/3) - (1/3)Q$
- Supply: $P = 2 + Q$

Example: Market equilibrium model

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Example: Market equilibrium model

- A market is said to be in the equilibrium if Qd = Qs.
- Substituting "Qd" equation into "Qs", we yield that

  $P = 2 + 10 - 3p$

  $4p = 12 \rightarrow p = 3.$

  $Q = 1.$

- Equilibrium output = $1 and Equilibrium price = 3 unit.

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Example: Market equilibrium model

- Qd = 30 - 5*income - 3P
- Qd = 10 - 3P
- So, we must have that

  30 – 5*income = 10

  Income = 4.

Example: Market equilibrium model

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Example: Market equilibrium model

- Income is $5 higher. That is, income is now $9.
- Demand equation is then given by,

  Qd = 30 - 5*9 - 3P

  Qd = - 15 – 3P

- Now solve for the equilibrium.

  P = 2 - 15 – 3P

  4p = -13 ➔ p = -13/4 ??? ➔ market no longer exists

- This is inferior goods. As income increases, demand curve shifts down.

Market equilibrium model

- The notion of market equilibrium is the notion for aggregate concepts.
- We look at the equilibrium as a whole, not by individualistic.
- So, when looking at the equilibrium, we have to use market demand and supply.
- So, we need to derive for market demand/supply, which can be done if we have information about individual demand/supply.

- A study has shown that there are three groups of Iphone users, namely, *crazy, love-it, and just-live-with-it.* Demand for Iphone of each group can be given by:

  crazy: Qc = 100 - p;

  Love-it: p = 50 - Ql;

  Just-live-with-it: Qj = 20 - p;

  – Find the domain set of price that justifies the demand equation for each group of Iphone user. And, rewrite each demand function in a more appropriate way.
  – At what domain set of prices, do all the three types of Iphone users stay active in the market?
  – Find the function for market demand for Iphone. Be precise about what is needed to make your equation justified.
- Suppose market supply equation is given by: $p = 4 + 3w + \frac{3}{8}Q$.
  – Find the equilibrium when w = 1/3 where w is wage rate for each unit of labor hired. How much does each type of consumer consume in the equilibrium?
  – What is the likely effect on market equilibrium when wage drops? State your prediction and develop intuition for your result. (Note: Answer to this question could be made in qualitative sense. You don't need to get into algebraic solution with numbers solved.)

Application: Restricting domain set for demand and the construction of market demand.

- Why this question?
  - Generally, domain/range set of linear function is

    R x R. Function is generally well-defined for the whole set of real number.
  - However, when applying linear function to express any mathematical relationship of economic variables, we need to impose some restrictions on the domain/range set that makes the relationship well-defined.

Application: Restricting domain set for demand and the construction of market demand.

crazy:	$Qc = 100 - p$;
Love-it:	$p = 50 - Ql$;
Just-live-with-it:	$Qj = 20 - p$;

- Price is greater than or equal to zero! Price cannot be negative!.
- Can price be $200? Would it make the three demand equations sensible? If no, what other conditions do we need to justify that p = 200 is possible in the reality?

Application: Restricting domain set for demand and the construction of market demand.

- $Crazy \Rightarrow \quad Q_c = 100 \ - p \quad ; \ p \leq 100.$

  $= 0 \qquad\qquad ; \ p > 100.$

- $Love - it \Rightarrow \quad Q_l = 50 \ - p \qquad ; \ p \leq 50.$

  $= 0 \qquad\qquad ; \ p > 50.$

- $live - with - it$

  $\Rightarrow \quad Q_j = 20 \ - p \qquad ; \ p \leq 20.$

  $= 0 \qquad\qquad ; \ p > 20.$

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Application: Restricting domain set for demand and the construction of market demand.

- Crazy is high-type consumer. Maximum willingness to pay is \$100. ($p \leq 100$)
- Just-live-with-it is easily attracted by other brands of smartphone, and thus only have the maximum willingness to pay equal to \$20. When price is above \$20, they choose to leave the market.
- Love-it group is in the middle between the two. They choose to stay if price is lower than \$50.

Application: Restricting domain set for demand and the construction of market demand.

- So, at what domain of prices, do all the three types stay active in the market?
- When p is less than $20, we have strict positive value of Q for each type of consumer.

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Application: Restricting domain set for demand and the construction of market demand.

- Market comprises of a number of individual consumers.
- Each person has its own individual demand.
- When looking at the equilibrium, we look at the outcome as a whole. Equilibrium has to be determined at the market level.
- So, we have to establish notion/definition of market demand.
- I'll show you two approaches for deriving market demand equation from given sets of individual demand functions.

Application: Restricting domain set for demand and the construction of market demand.

- Definition
  - Market demand = summing up Q for each individual at each given price.
  - Different prices ➔ different level of each individual Q combined.

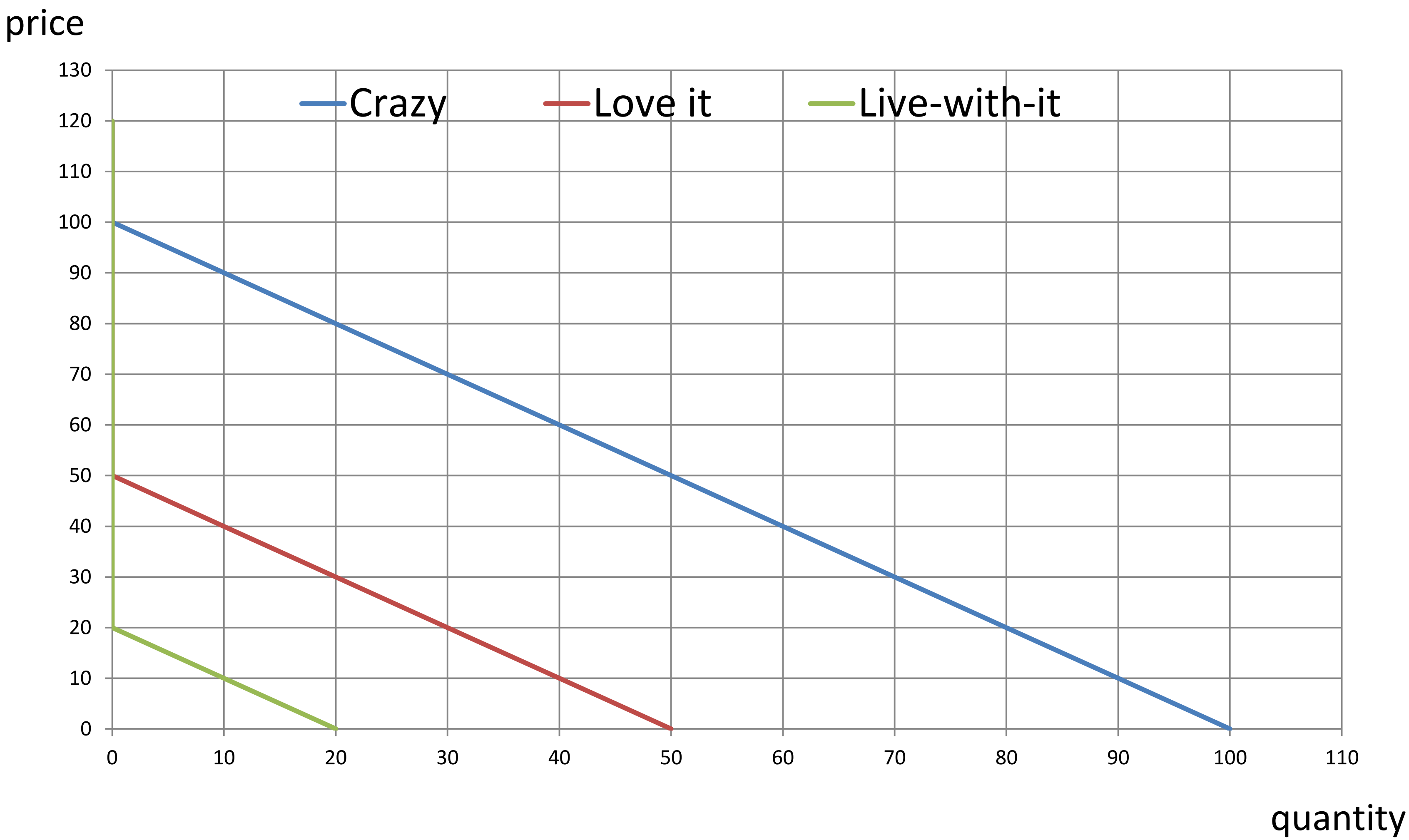
Application: Restricting domain set for demand and the construction of market demand.

- Graphical approach, constructing market demand curve:
  - This is called the horizontal sum for each individual demand curve.
  - Putting all the curves of individual demand into the same figure, sum-up Q for each individual at each given price, label all the kinked points (if any), and then derive equation for each portion of the demand curve.

Application: Restricting domain set for demand and the construction of market demand.

- As individual demand curve is downward sloping, market demand curve should preserve this characteristic as well. That is, as price decreases, total Q combined increases
- Typically, Market demand equation for the derived demand curve will be a ***step*** function.

crazy:	$Q_c = 100 – p$; ➔	$p = 100 - Q_c$
Love-it:	$p = 50 - Q_l$; ➔	ok!
Just-live-with-it:	$Q_j = 20 - p$; ➔	$p = 20 - Q_j$



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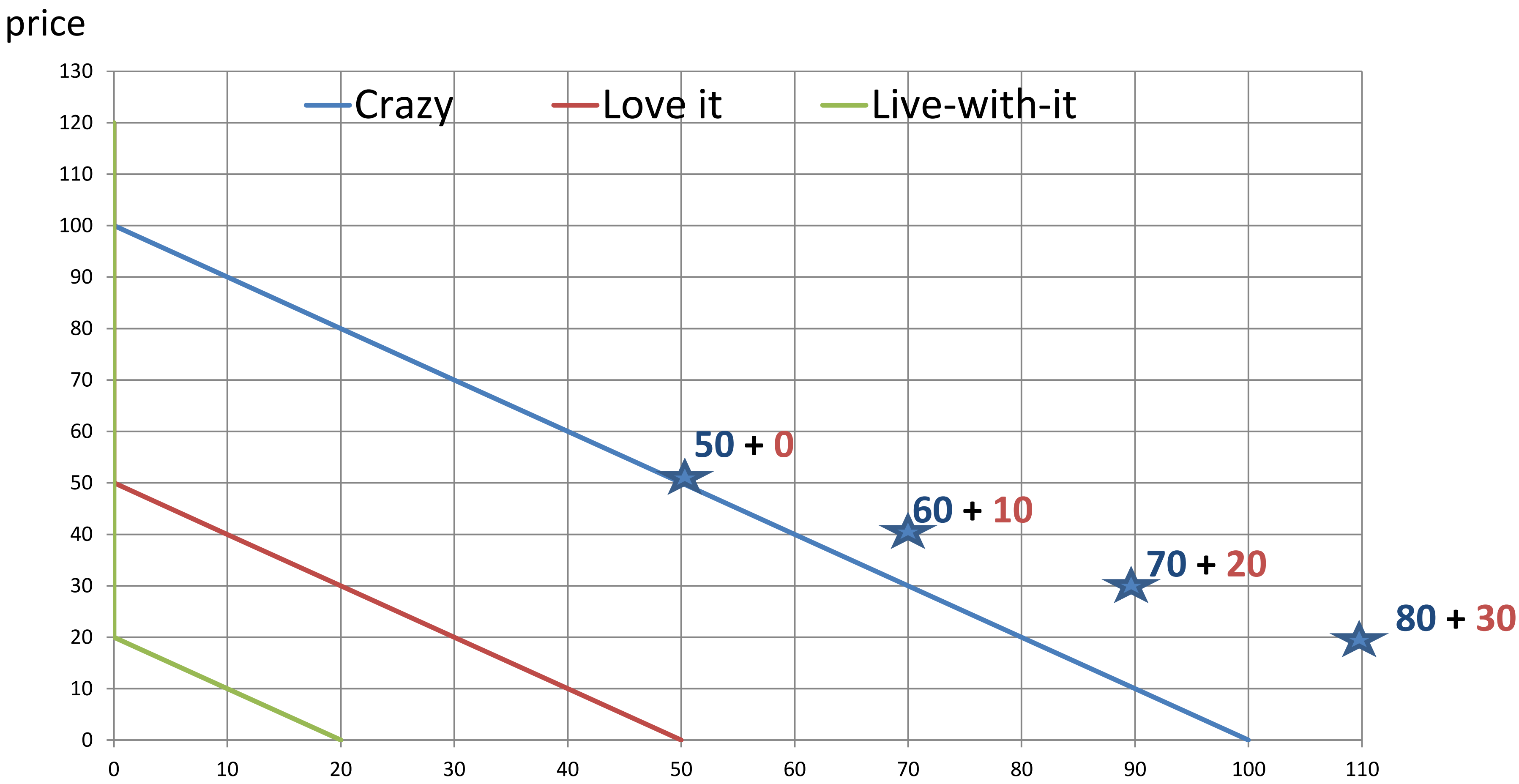


When price = $10, total demand = **90** + **40** + **10** = **140!**

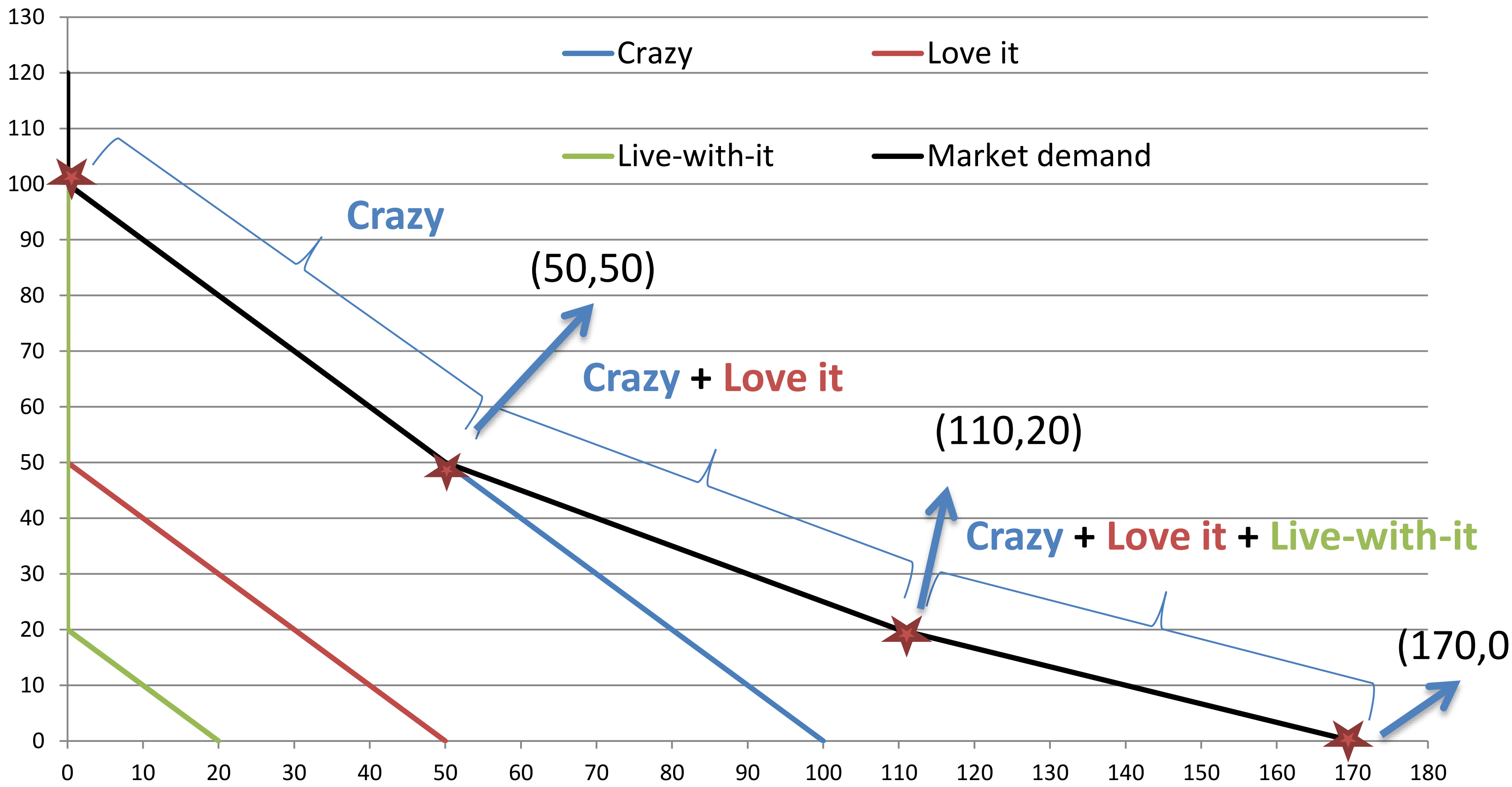
Demand schedule

Price/unit	Crazy	Love it	Just-live-with-it	Q-combined
120	0	0	0	0
110	0	0	0	0
100	0	0	0	0
90	10	0	0	10
80	20	0	0	20
70	30	0	0	30
60	40	0	0	40
50	50	0	0	50
40	60	10	0	70
30	70	20	0	90
20	80	30	0	110
10	90	40	10	140
0	100	50	20	170

Price decreases

Quantity Increases

Market demand curve = *horizontal* sum of individual demand curve

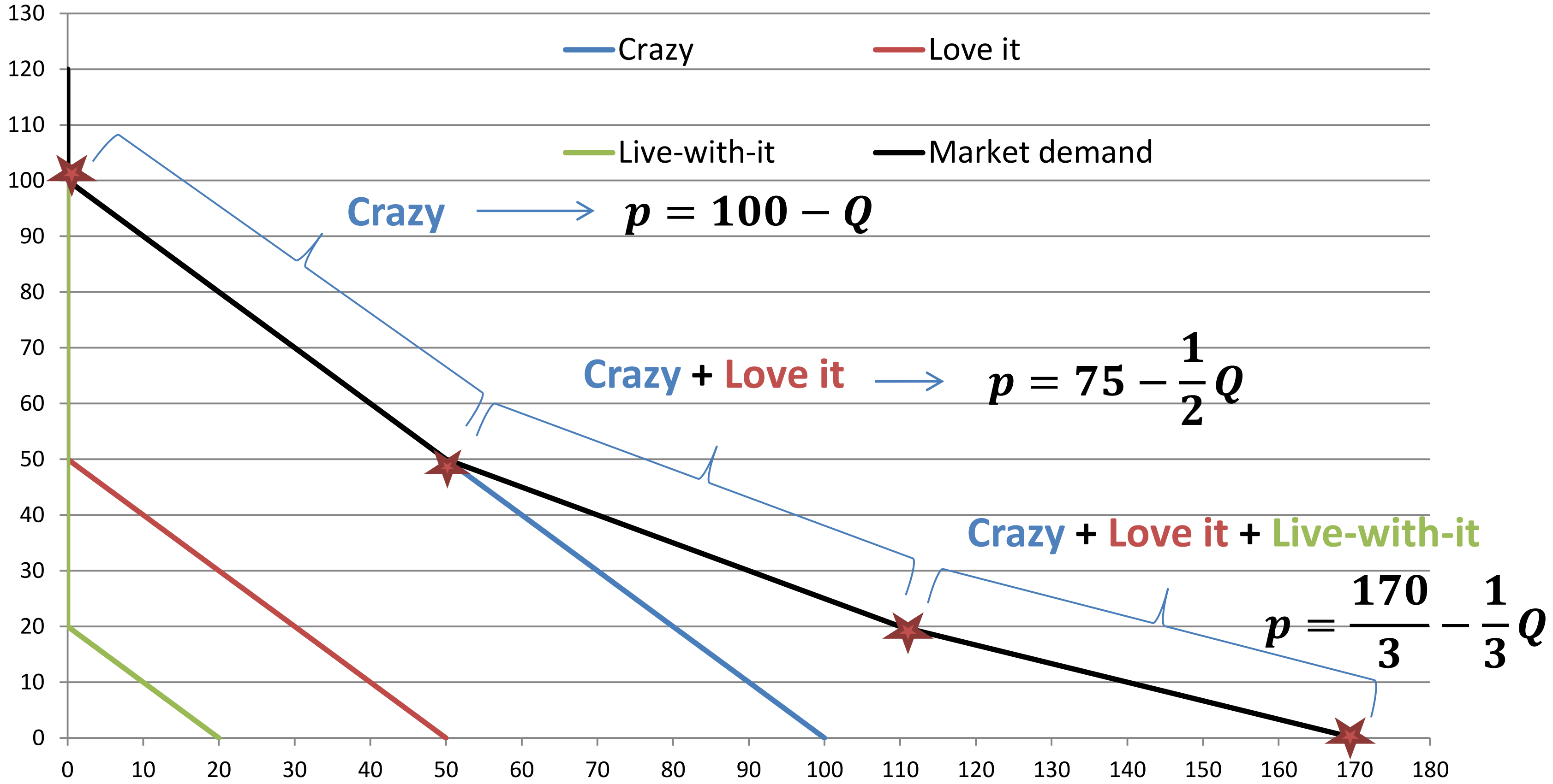


Demand curve flatters out as quantity of output consumed increases!

How to find the market demand equation?

- Find an equation for each part of the demand curve.
  - As we have all the important points for each part of the demand curve, we can find the equation that describe demand curve for each part.
- For example, we know that slope of the second part of demand curve (crazy + love-it) would be equal to -0.5.
  - We knew that (50,50) is on the demand curve.
- To find the equation, apply point-slope formula for derive the equation.

Market demand curve = *horizontal* sum of individual demand curve



Demand curve flatters out as quantity of output consumed increases!

Equation for market demand curve

- $$\boldsymbol{p = 100 - Q \quad ; \qquad 0 \le Q \le 50}$$
  $$\boldsymbol{= 75 - \frac{1}{2}Q \quad ; \qquad 50 < Q \le 110}$$
  $$\boldsymbol{= \frac{170}{3} - \frac{1}{3}Q \quad ; \qquad 110 < Q \le 170}$$
- Slope value getting smaller as Q increases.
- Tedious.. Several steps, but graphically

Illustration makes it very intuitive.

Application: Restricting domain set for demand and the construction of market demand.

- Mathematical summation! Just a simple direct summation.
- But, you have to be very careful when using this method. Not all individual will stay active at the considered price.
- Make sure that you don't sum up Q-equation without checking if that individual remains active in the market.

Application: Restricting domain set for demand and the construction of market demand.

- $Market\ demand = Q = Q_c + Q_l + Q_j$
- $Q = \mathbf{100 - p} + \mathbf{50 - p} + \mathbf{20 - p} = 170 - 3p$
  - A naïve guy who doesn't make a careful consideration would answer that the above equation is the mathematical representation of market demand.
- But, wait!!! But, this is not a step function.
  - We've seen before that market demand curve doesn't have the same slope throughout the entire curve, i.e. flatter out as more of output consumed.
- So, what's wrong?

Application: Restricting domain set for demand and the construction of market demand.

- You were summing Q for some domain sets that result in negative value of quantity. This doesn't make sense in economics!
- For example, if price is $60, demand of love-it group would be 0. If you were just summing up that way, you would add "50 – 60 = -10" into the equation. This is wrong!

Application: Restricting domain set for demand and the construction of market demand.

- Split the domain set of prices into different regions, each associated with what types of consumer are staying in the market.
- $Q = 100 - p \quad ; \quad 100 \le p \le 50$

  $= 150 - 2p \quad ; \quad 50 < p \le 20$

  $= 170 - 3p \quad ; \quad p < 20$

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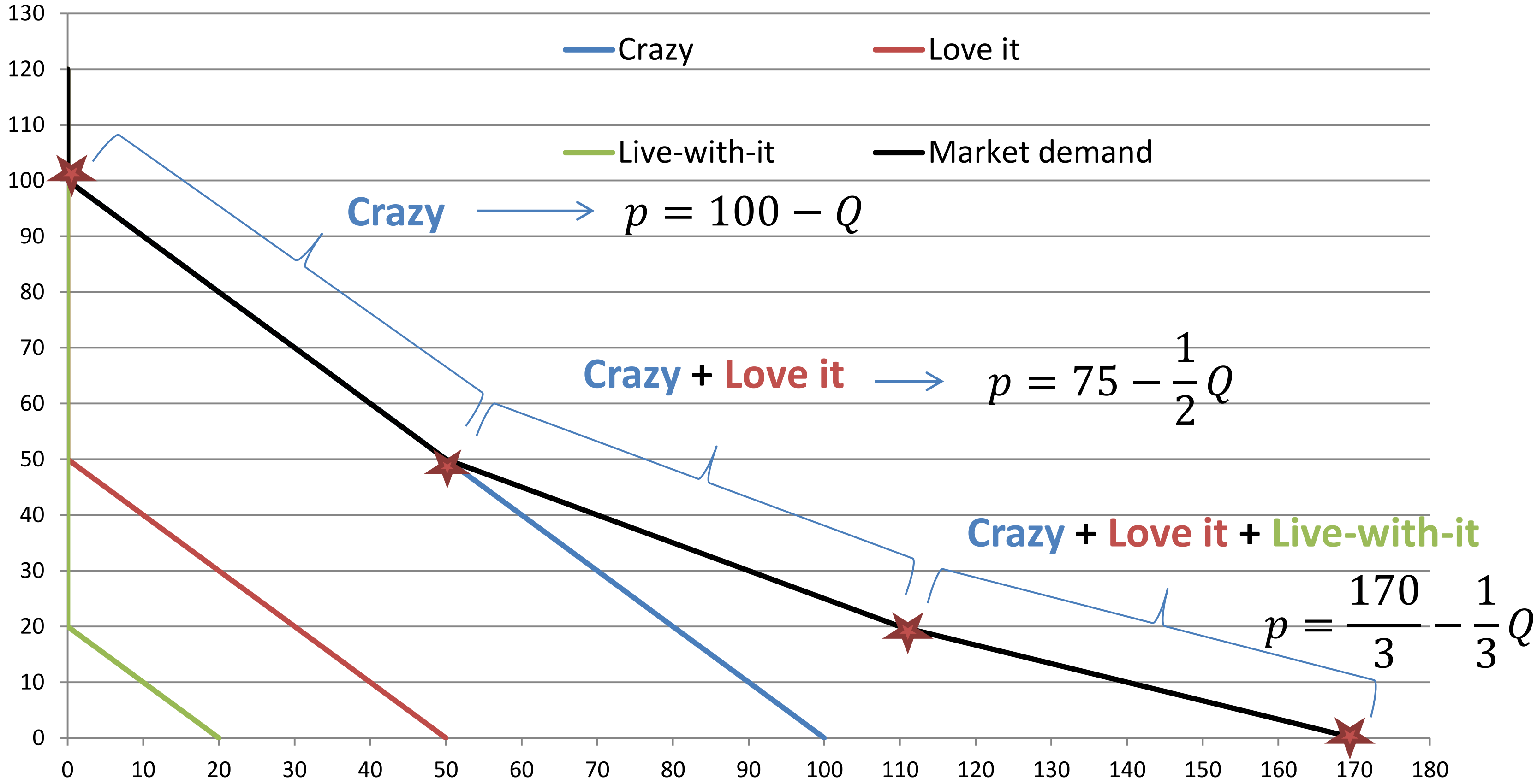
Market demand = market supply?



Demand curve flatters out as quantity of output consumed increases!

For $W = \frac{1}{3}$, supply equation is then…

First try..

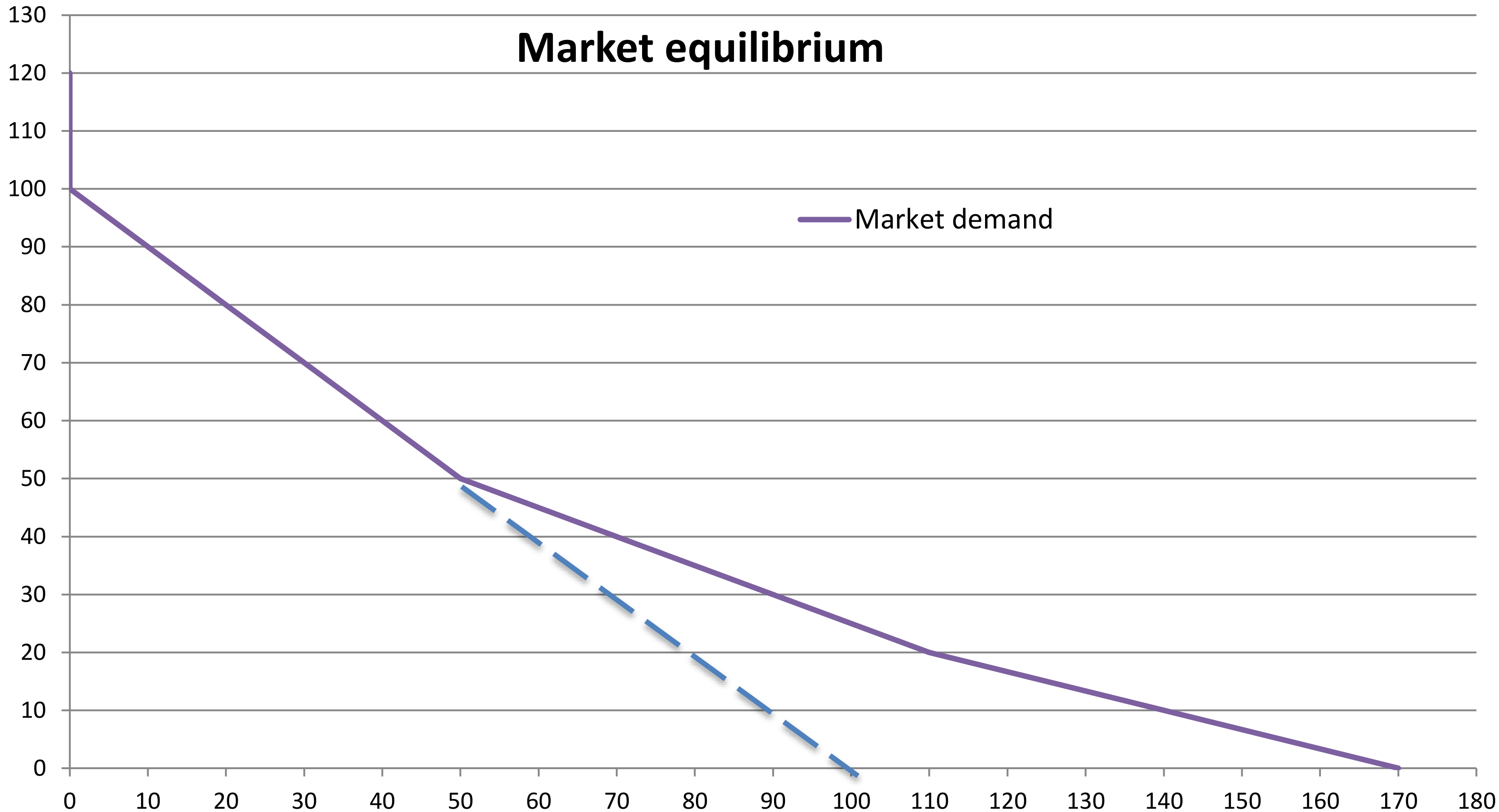
- Let's conjecture that supply curve intersects with demand curve in _______________.

- Is this right? Why?

Second try ….

- Let's recalculate by supposing that supply curve intersects with demand curve in the ________________.

Graphically illustrated!



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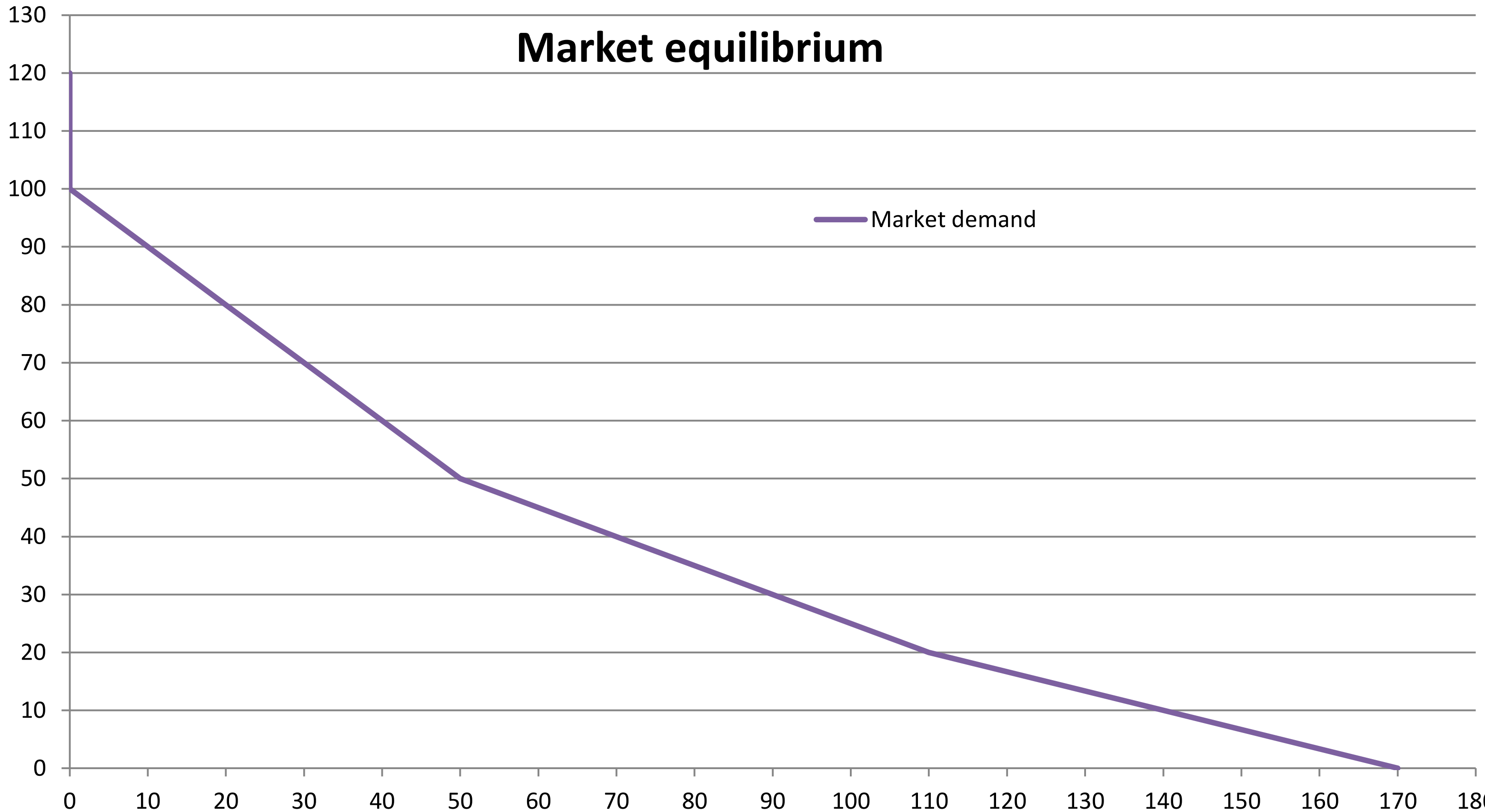
For W = $\frac{1}{3}$, supply equation is then…

- An increase in “w” represents the so called a ____________ “supply shock”.
- A decrease in “w” represents the so called a ____________ “supply shock”.

Graphically illustrated!



Introducing taxation

- Effect of unit tax: market allocation, tax burden.
  - Market allocation: quantity change?
  - Tax burden: how much consumer’s paying for the tax? Does it matter if tax is actually being imposed on producer?

Introducing taxation

- Two different concepts of tax burden, i.e. who's paying the tax.
  - Legal burden: who has the duty to bring the money to IRS.
  - Economics burden: how to share the tax cost between consumers and producers.
- What we know is that "*economic incidence would be identical regardless to where tax has been imposed*."
  - The idea is that person who has legal burden will just try to pass on tax burden to the other party. The more limited choice you have for the adjustment, the higher burden you have to bear in the equilibrium.

Introducing taxation

- With tax, price that consumer's paying will not be equal to price that producer's receiving.
- So, we need to make it clear from now that
  - $p^s = price\ that\ producer\ gets$
  - $p^d = price\ that\ consumer\ pays$
  - $p^s \neq p^d$ ➔ price gap *or* price wedges (tax wedges)
- Behavior of buyers and suppliers would focus at their perceived prices.

Introducing taxation

- Suppose that:
  - Demand: $p^d = a - bQ^d \quad ; \;\; a \geq 0, \quad b \leq 0.$
  - Supply : $p^s = c + dQ^s \quad ; \;\; d \geq 0.$
- Without tax, we solve for equilibrium by imposing that $Q^d = Q^s$.
- This would have been the same as we impose that $p^d = p^d$ since they both have the same perceived price.

Introducing taxation

- As said before, when tax is introduced, government creates price gap.
- So, solving for equilibrium in the market, with tax imposed, would make sense only when (i) we impose the market clearing condition, i.e. $Q^d = Q^s$, or states the condition that explains to you about how p^d and p^s are related under the equilibrium.

Introducing taxation

- The relationship between p^d and p^sdepends on how tax is being imposed.
  - If the tax is imposed on consumer, it would be that: $p^d = p^s + t$.
  - If the tax is imposed on producer, it would be that: $p^s = p^d - t$.

Example

- Suppose that demand/supply equation can be given by
  - $P^d = 20 - Q^d$
  - $P^s = 10 + Q^s$

  a. Find the market equilibrium.
  b. Suppose that government imposes tax on producer equal to $4 per unit, determine market equilibrium under taxation.
  c. Redo the same exercise, but now suppose that tax is imposed on consumer for the same amount.
  d. How much is the tax revenue that the government can collect in the equilibrium?
  e. Calculate the tax burden.

Tax on producer

Demand: $p^d = a - bQ^d \quad ; \; a \geq 0, \quad b \leq 0.$

Supply : $p^s = c + dQ^s \quad ; \; d \geq 0.$

- $p^s = p^d - t$
- $p^s = p^d - t = c + dQ^s$
- $a - bQ^d - t = c + dQ^s$
- $Q^D = Q^S = Q^*$
- $Q^* = \frac{a-c-t}{b+d} = \frac{a-c}{b+d} - \frac{t}{b+d}$
- Market size contracts after the taxation.

Tax on producer

- Consumer's paying more
  - $\Rightarrow p^d = \frac{ad+bc}{b+d} + \frac{b}{b+d}t$
- Producer's getting less
  - $\Rightarrow p^s = \frac{ad+bc}{b+d} - \frac{d}{b+d}t$
- Say t = $1, this means,
  - Fraction that consumers pay is $\frac{b}{b+d}$ cents.
  - Fraction that producers pay is $\frac{d}{b+d}$ cents.

Who pays more?

- As demand becomes steeper, consumer's paying more!
  - Think about necessity products! The extreme cases is when "b = *infinitiy*". That is, demand curve is vertical line. Under this case, consumer takes all 100% of tax burden.
- As supply becomes steeper, producer's paying more relatively.
  - Firm finds it difficult to adjust scale of the production.
- Main Idea: Whoever cannot instantaneously adjust themselves to the tax imposed would unfortunately have to bear a larger portion of the tax burden.

A thought experiment question: What about subsidy?

- What do we know about the effect of subsidy?
- Do we have to start everything from the beginning to derive the implications of subsidy program for market equilibrium?

Multimarket Equilibrium

- So far, we have only analyzed the market equilibrium concept when we assume that there is only single market in the economy.
- In fact, economy comprises of so many markets.
- Equilibrium concept that we use so far can be extended to the more realistic environment where we have N markets.

Example

- Let demand and supply of market for goods A and B be

$$Da = 20 - 3Pa - 2Pb$$

$$Sa = 12 + 2Pa + 5Pb$$

$$Db = 4 - Pa - 3Pb$$

$$Sb = -1 + 2Pa + 4Pb$$

Answer the following questions:

- State equilibrium conditions of this multimarket model.
- Find the equilibrium price and quantity of the market for goods A and the equilibrium price and quantity of the market for goods B.