

Chapter 4 Integration

Section 4.1 Antiderivatives

Definition: A function F is called an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all x in I .

For instance, let $f(x) = x^2$. We know that if $F(x) = (1/3)x^3$, then $F'(x) = x^2 = f(x)$. But the function $G(x) = (1/3)x^3 + 75$ also satisfies $G'(x) = x^2$. Therefore, both F and G are antiderivatives of f . Indeed, any function of the form $H(x) = (1/3)x^3 + C$, where C is a constant, is an antiderivative of f .

Thus, if F and G are any two antiderivatives of f , then $F' = f(x) = G'(x)$ so $G(x) - F(x) = C$, where C is a constant. We can write this as $G(x) = F(x) + C$, so we have the following result.

Theorem: If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + c$$

where c is an arbitrary constant.

Table of Antidifferentiation Formulas

Function	Particular antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n (n \neq -1)$	$(x^{n+1})/(n+1)$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

Note : To obtain the most general antiderivative from the particular ones, we have to add a constant (or constants).

Example 1: Find all functions f such that $f'(x) = 8x^3 + 12x + 3$.

Example 2: Find f where $f'(x) = 8x^3 + 12x + 3$, $f(1) = 6$.

Section 4.2 The Definite Integral

Definition of a Definite Integral :

If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(=a), x_1, x_2, \dots, x_n(=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

Because we have assumed that f is continuous, it can be proved that the limit in the definition above always exists and gives the same value no matter how we choose the sample points x_i^* . If we take the sample points to be right endpoints, then $x_i^* = x_i$ and the definition of an integral becomes

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x$$

If we choose the sample points to be left endpoints, then $x_i^* = x_{i-1}$ and the definition becomes

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i-1})\Delta x$$

Alternatively, we could choose x_i^* to be the midpoint of the subinterval or any other number between x_{i-1} and x_i .

Note 1: The symbol $\int$ was introduced by Leibniz and is called an integral sign. It is an elongated S and was chosen because an integral is a limit of sums. In the notation

$\int_a^b f(x)dx$, $\int f(x)dx$ is called the **integrand** and a and b are called the **limits of integration**; a

is the **lower limit** and b is the **upper limit**. The symbol dx has no official meaning by itself;

$\int_a^b f(x)dx$ is all one symbol. The procedure of calculating an integral is called integration.

Note 2: The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on x . In fact, we could use any letter in place of x without changing the value of the integral.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Note 3: The sum

$$\sum_{i=1}^n f(x_i^*)\Delta x$$

that occurs in the definition above is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826-1866). We know that if f happens to be positive, the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see figure 4.1(a)). By comparing the definition above with the definition of area, we see that the definite integral

$\int_a^b f(x)dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b . (See figure 4.1(b)).

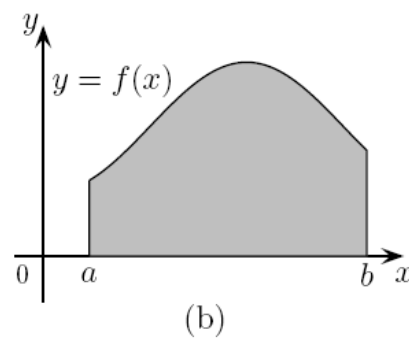
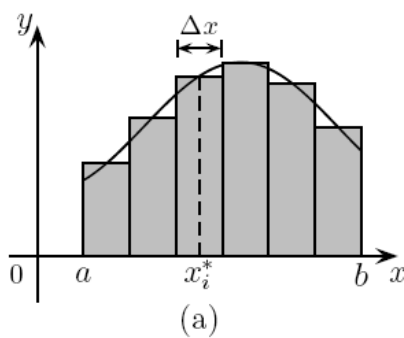
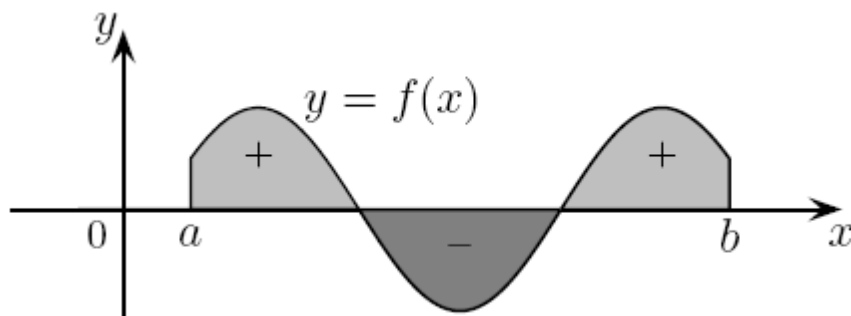


Figure 5.1:

If f takes on both positive and negative values then the Riemann sum is the sum of the areas of the rectangles that lie above the x -axis and the *negatives* of the areas of the rectangles that lie below the x -axis. When we take the limit of such Riemann sums, we get the situation illustrated in figure below.



A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x -axis and below the graph of f , and A_2 is the area of the region below the x -axis and above the graph of f .

Note 4: Although we have defined $\int_a^b f(x)dx$ by dividing $[a,b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

Properties of the Definite Integral

$$(1) \int_b^a f(x)dx = - \int_a^b f(x)dx$$

$$(2) \int_a^a f(x)dx = 0$$

Properties of the Integral

$$(1) \int_a^b cdx = c(b-a) \text{ where } c \text{ is any constant}$$

$$(2) \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$(3) \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$(4) \int_a^b cf(x)dx = c \int_a^b f(x)dx \text{ where } c \text{ is any constant}$$

$$(5) \int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Comparison Properties of the Integral

$$(6) \text{If } f(x) \geq 0 \text{ for } a \leq x \leq b, \text{ then } \int_a^b f(x)dx \geq 0$$

$$(7) \text{If } f(x) \geq g(x) \text{ for } a \leq x \leq b,$$

$$\text{then } \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

$$(8) \text{If } m \leq f(x) \leq M \text{ for } a \leq x \leq b,$$

$$\text{then } m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Section 4.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part I (FTC I)

If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on $[a,b]$ and differentiable on (a,b) , and $g'(x) = f(x)$.

The Fundamental Theorem of Calculus, Part II (FTC II)

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f , that is, $F'(x) = f(x)$

Basically,

Part I says “derivatives” and “integrals” are opposite (or inverse operations).

Part II gives us (finally) a method to evaluate definite integrals without using the limit.

Example 1: Find the derivative of the function.

$$1. g(x) = \int_0^x \cos t \, dt \quad 2. g(x) = \int_0^x \sin^2 t \, dt$$

$$3. g(t) = \int_0^t x^3 \, dx \quad 4. g(x) = \int_0^{x^4} \sec t \, dt$$

$$5. y = \int_0^{\cos x} (1 + \sin t) \, dt \quad 6. y = \int_{1/x^2}^0 \sin^3 t \, dt$$

Example 2: Evaluate the integral.

$$1. \int_1^2 (x^2 - x^3) dx$$

$$2. \int_{-\pi/2}^{\pi/2} \cos x dx$$

$$3. \int_0^1 (2x - 1)^2 dx$$

$$4. \int_{-2}^5 6 dx$$

$$5. \int_{-2}^3 (x^{-4}) dx$$

$$6. \int_0^{\pi/6} \csc \theta \cot \theta d\theta$$

$$7. \int_{-\pi}^{\pi} f(x) dx \quad \text{where} \quad f(x) = \begin{cases} x & \text{if } -\pi \leq x \leq 0 \\ \sin x & \text{if } 0 < x \leq \pi \end{cases}$$

$$8. \int_0^{\pi/3} \frac{1 - \sin^2 x}{\cos^4 x} dx$$

Section 4.4 Indefinite Integrals

Recall :

$$\text{FTC I : } \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

$$\text{FTC II : } \int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

The notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an indefinite integral. Thus $\int f(x)dx = F(x)$ means $F'(x) = f(x)$

Note: A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function (or family of functions).

Table of Indefinite Integral

$$(1) \int cf(x)dx = c \int f(x)dx$$

$$(2) \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$(3) \int kdx = kx + C$$

$$(4) \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$(5) \int \sin x dx = -\cos x + C$$

$$(6) \int \cos x dx = \sin x + C$$

$$(7) \int \sec^2 x dx = \tan x + C$$

$$(8) \int \csc^2 x dx = -\cot x + C$$

$$(9) \int \sec x \tan x dx = \sec x + C$$

$$(10) \int \csc x \cot x dx = -\csc x + C$$

$$(11) \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$(12) \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$(13) \int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + C$$

Example : Evaluate

$$1. \int \sqrt[3]{x} dx \quad 2. \int \frac{\sin 2x}{\sin x} dx \quad 3. \int_1^2 \frac{y + 5y^7}{y^3} dy \quad 4. \int_0^{\pi/3} \frac{\sin x + \sin x \tan^2 x}{\sec^2 x} dx \quad 5. \int_0^{3\pi/2} |\sin x| dx$$

Section 4.5 Integration by Substitution

Section 4.6 Evaluating Definite Integrals by Substitution

Consider the function $f(x) = 5(x^2 + 3x)^4(2x + 3)$. To find the antiderivative $F(x)$ we want $F'(x) = f(x)$, it looks as if the Chain Rule was applied to $F(x)$ to find $f(x)$. Try to reverse it to get $F(x) = (x^2 + 3x)^5 + C$. Then $F'(x) = 5(x^2 + 3x)^4(2x + 3) = f(x)$.

What about $f(x) = 3x \sin(x^2)$, $F(x) = ?$

An initial guess might be $F(x) = -\cos(x^2) + C$, $F'(x) = 2x \sin(x^2) \neq f(x) !!!$

What is missing is a factor of $3/2$, however since constant multiplies are not affected by integration, we can simply adjust after the fact.

$$F(x) = -(3/2)\cos(x^2) + C$$

$$F'(x) = (-3/2)(2x)(-\sin(x^2)) = 3x \sin(x^2) = f(x)$$

This technique enables us to integrate function via a “reverse chain rule” by involving a new variable which the integral is re-written.

Example: $\int x^2 \sin(x^3) dx = ?$

- First check that it does look like a reverse chain rule. ie. Look if the derivative of the “inside” appears on the “outside” (apart from a multiplicative constant).
- Then substitute “ u ” as the entire inside function.

Here $u = x^3$, we have to have everything as one variable, so we find $du/dx = 3x^2 \rightarrow du = 3x^2 dx$

Now substitution

$$\int x^2 \sin(x^3) dx = \int \sin(x^3) x^2 dx = \int \sin(u) \frac{1}{3} du = \frac{1}{3} \int \sin(u) du = -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos x^3 + C$$

The Substitution Rule:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 1: Find

1. $\int \frac{\sin(t^{-2} + 1)}{t^3} dt$

2. $\int y^3 \sqrt{2y^4 - 1} dy$

The Substitution Rule for Definite Integral:

If $g'(x)$ is continuous on $[a,b]$ and f is continuous on the range $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 2: Find $\int_0^{\pi/2} \sin x \cos x dx$

Example 3: Find

$$1. \int_2^4 \frac{dx}{(2x-3)^2}$$

$$2. \int_0^4 \frac{dx}{(2x-3)^2}$$

$$3. \int_0^7 \sqrt{4+3x} \, dx$$

$$4. \int_{1/6}^{1/2} \csc(\pi t) \cot(\pi t) dt$$

$$5. \int_0^{\pi/2} \cos x \sin(\sin x) dx$$

$$6. \int \sin^3 x \cos x dx$$

$$7. \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$$

$$8. \int x^3 (x^2 + 1)^{99} dx$$

$$9. \int \frac{1}{\sqrt{1-4x^2}} dx$$

$$10. \int \frac{1}{x^2 + a^2} dx$$

$$11. \int \frac{x}{x^4 + 9} dx$$