

Differentiation: Application

- Related Rate
- L'Hôpital's Rule

1 Related Rate

The derivative $\frac{dy}{dx}$ of a function $y = f(x)$ is its instantaneous rate of change with respect to variable x .

- Suppose a function $s(t)$ represent the position o an object moving on a horizontal or vertical line. Then the **time rate of change** $\frac{ds}{dt}$ is the velocity of the object.
- In general, a time rate of change tells how fast the quantity is changing.

Guidelines for Solving Related Problems

- Carefully read the problem. Draw a picture if possible.
- Label with symbols all quantities that change with time.
- Write down all the rates that are given. Using derivative notation, write down the rate that you want to find.
- Set up an equation or a function that relates all the variables you have introduced.
- Differentiate the equation or function found in step (iv) with respect to t . This may need to use **implicit differentiation**. The resulting equation after differentiation rates at which the variables change with time.

Example 1.1. A spherical balloon is expanding with time. How is the rate at which the volume increases related to the rate at which the radius increases?

Solution: Let V be the volume of a sphere with radius r . Then

Example 1.2. Air is being pumped into a spherical balloon at a rate of $20 \text{ ft}^3/\text{min}$. At what rate is the radius changing when the radius is 3 ft ?

[Ans: $\frac{5}{9\pi} \text{ ft/min}$]

Solution:

Example 1.3. A woman jogging at a constant rate of 10 km/h crosses a point P heading north. Ten minutes later a man jogging at a constant rate of 9 km/h crosses the same point heading east. How fast is the distance between the joggers changing 20 minutes after the man crosses P ? [Ans: $\frac{77}{\sqrt{34}} \text{ km/hr}$]

Solution: Let x and y be the locations of the man and the women from point P , respectively. Let z be the distance between the man and the woman. Then by the **Pythagorean theorem**, $z^2 = x^2 + y^2$.

Example 1.4. A light is on the top of a 12 ft tall pole and a 5.5ft tall person is walking away from the pole at a rate of 2 ft/sec.

- (a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?
- (b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

[Ans: 48/13, 22/13 ft/sec]

Solution:

Let x be the distance of the tip of the shadow from the pole,

x_p be the distance of the person from the pole and

x_s be the length of the shadow.

2 L'Hôpital's Rule

This section uses derivative to calculate certain limits with indeterminate forms.

Definition 2.1 (Indeterminate Form). The limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

has the **indeterminate form** $\frac{0}{0}$ at $x = a$ if

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0,$$

and has the **indeterminate form** $\frac{\infty}{\infty}$ at $x = a$ if

$$\lim_{x \rightarrow a} |f(x)| = \infty \quad \text{and} \quad \lim_{x \rightarrow a} |g(x)| = \infty,$$

i.e. either (i) $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$ or

(ii) $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = -\infty$, or

(iii) $\lim_{x \rightarrow a} f(x) = -\infty$ and $\lim_{x \rightarrow a} g(x) = +\infty$, or

(iv) $\lim_{x \rightarrow a} f(x) = -\infty$ and $\lim_{x \rightarrow a} g(x) = -\infty$.

Note that the definition for indeterminate form above also holds for the one-sided limit and the limit at infinity:

$$x \rightarrow a^-, \quad x \rightarrow a^+, \quad x \rightarrow -\infty, \quad x \rightarrow \infty.$$

Theorem 2.1 (L'Hôpital's Rule). Suppose f and g are differentiable functions on an open interval containing the number a , except a itself, and that $g'(x) \neq 0$ for all x in the interval, except possibly at a .

Suppose $\lim_{x \rightarrow a} f(x)/g(x)$ is an indeterminate form and $\lim_{x \rightarrow a} f'(x)/g'(x) = L$ or $\pm\infty$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Example 2.1. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$.

This limit has the indeterminate form $0/0$ at $x = 0$, because

So we can apply L'Hôpital's rule as follows.

Example 2.2. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(x)}{e^x - e^{-x}}$. [Ans: 1/2]

Solution: This limit has the indeterminate form $\frac{0}{0}$ at $x = 0$. So we can apply L'Hôpital's rule as follows.

Example 2.3. Evaluate $\lim_{x \rightarrow \infty} \frac{\ln(x)}{e^x}$. [Ans: 0]

Solution: This limit has the indeterminate form $\frac{\infty}{\infty}$, because

So we can apply L'Hôpital's rule as follows.

Example 2.4. Evaluate $\lim_{x \rightarrow \infty} \frac{6x^2+5x+7}{4x^2+2x}$ by applying L'Hôpital's rule. [Ans: 0]

Solution: This limit has the indeterminate form $\frac{\infty}{\infty}$. This problem has to successively apply L'Hôpital's rule.

Example 2.5. Evaluate $\lim_{t \rightarrow \frac{\pi}{2}^+} \frac{\tan(t)}{\tan(3t)}$. [Ans: 3]

Solution: This problem has to successively apply L'Hôpital's rule. This given limit has the indeterminate form $\frac{\infty}{\infty}$ because

[Hint: Use $\sin(2A) = 2 \sin(A) \cos(A)$]

Example 2.6 (Exercise). Evaluate $\lim_{x \rightarrow 1^+} \frac{\ln(x)}{\sqrt{x-1}}$. [Ans: 0]

Solution: This limit has the indeterminate form

2.1 Other Indeterminate Forms

We will consider additional indeterminate forms:

$$\infty - \infty, \quad 0 \cdot \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty,$$

which can be rearranged to be in the indeterminate form, either in $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 2.7. Evaluate $\lim_{x \rightarrow 0^+} \left[\frac{3x+1}{\sin(x)} - \frac{1}{x} \right]$. [Ans: 3]

Solution: This limit has the indeterminate form

Rearranging by using the common denominator will give the indeterminate form $\frac{0}{0}$.

Example 2.8 (Exercise). Evaluate $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right)$. [Ans: 1]

Solution: This limit has the indeterminate form $0 \cdot \infty$. Notice that we can write

$$x \sin\left(\frac{1}{x}\right) = \frac{\sin(1/x)}{1/x}.$$

The indeterminate forms: 0^0 , ∞^0 , 1^∞

Suppose $\lim_{x \rightarrow a} f(x)^{g(x)}$ is in the form 0^0 , ∞^0 or 1^∞ . Let $y = f(x)^{g(x)}$.

$$\begin{aligned}\ln(y) &= \ln(f(x)^{g(x)}) \\ \ln(y) &= g(x) \ln(f(x)) \\ \lim_{x \rightarrow a} \ln(y) &= \lim_{x \rightarrow a} g(x) \ln(f(x))\end{aligned}$$

will be in the indeterminate form $0 \cdot \infty$ (why?). Suppose the L'Hôpital's rule is applied to obtain

$$L = \lim_{x \rightarrow a} g(x) \ln(f(x))$$

and using the continuity of function $\ln(\cdot)$:

$$\ln\left(\lim_{x \rightarrow a} y\right) = \lim_{x \rightarrow a} \ln(y) = \lim_{x \rightarrow a} g(x) \ln(f(x)) = L.$$

That is,

$$\ln\left(\lim_{x \rightarrow a} y\right) = L \quad \Rightarrow \quad \lim_{x \rightarrow a} y = e^L.$$

Note that the process above also holds for the one-sided limit and the limit at infinity:

$$x \rightarrow a^-, \quad x \rightarrow a^+, \quad x \rightarrow -\infty, \quad x \rightarrow \infty.$$

Example 2.9. Evaluate $\lim_{x \rightarrow 0} x^{1/\ln(x)}$

[Ans: e]

Solution:

Note that $\lim_{x \rightarrow 0} \ln(x) = -\infty$. So $\lim_{x \rightarrow 0} x^{1/\ln(x)}$ is in the form 0^0 .

Example 2.10 (Exercise). Evaluate $\lim_{x \rightarrow \infty} \left(1 - \frac{3}{x}\right)^{2x}$

[Ans: e^{-6}]

Solution: Note that this is in the form 1^0 .