Group 6

Question 0: (required for all)

0.1) Given that $Z = \frac{x^3 - y^3}{x^2 y^2}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -Z$

0.2) Given that $Z = \frac{x - y}{x + y}$, use the total differential and calculate the change in Z when x= 1 and y =1. What would happen to Z if X increases by 2 units while Y decreases by 2 units?

0.3) If z = $2x^2y + 3xy + y^2$ where $x = r^2 + 2rs$ and $y = 2r - 4s$, then by means of the chain rule, (0.3a) find ∂z/∂s and ∂z/∂r ; (0.3b) evaluate when r = 1 and s = 0

0.4) For $2x^2 + 3y^2 + 2z^2 = 16$, evaluate ∂z/∂y when x = 1, y = 2, z = -1.

0.5) Given that $ln\,(x + y + z) + xyz = ze^{x+y+z}$, evaluate ∂z/∂x when x = 0, y = 1, z = 0.

0.1)

$$\frac{\partial z}{\partial x} = \frac{x^2y^2\frac{\partial(x^3-y^3)}{\partial x} - (x^3-y^3)\frac{\partial(x^2y^2)}{\partial x}}{(x^2y^2)^2}$$

$$= \frac{x^2y^2(3x^2) - (x^3-y^3)(2xy^2)}{(x^2y^2)^2} = \frac{3x^4y^2 - 2x^4y^2 + 2xy^5}{x^4y^4} = \frac{xy^2(3x^3-2x^3+2y^3)}{x^4y^4} = \frac{x^3+2y^3}{x^3y^2}$$

$$\frac{\partial z}{\partial y} = \frac{x^2y^2\frac{\partial(x^3-y^3)}{\partial y} - (x^3-y^3)\frac{\partial(x^2y^2)}{\partial y}}{(x^2y^2)^2}$$

$$= \frac{x^2y^2(-3y^2) - (x^3-y^3)(2x^2y)}{(x^2y^2)^2} = \frac{-3x^2y^4 - 2x^5y + 2x^2y^4}{x^4y^4} = \frac{x^2y(-3y^3-2x^3+2y^3)}{x^4y^4} = \frac{-2x^3-y^3}{x^2y^3}$$

$$x\frac{\partial z}{\partial x} = x\left(\frac{x^3+2y^3}{x^3y^2}\right) = \frac{x^3+2y^3}{x^2y^2}$$

$$y\frac{\partial z}{\partial y} = y\left(\frac{-2x^3+5y^3}{x^2y^3}\right) = \frac{-2x^3-y^3}{x^2y^2}$$

$$\therefore\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^3+2y^3}{x^2y^2} + \left(\frac{-2x^3-y^3}{x^2y^2}\right)$$

$$= -\frac{x^3+y^3}{x^2y^2} = -z$$

0.2)

$$dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy = 0$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)\frac{\partial (x-y)}{\partial x} - (x-y)\frac{\partial (x+y)}{\partial x}}{(x+y)^2}$$

$$= \frac{(x+y)(1) - (x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2} > 0 \ (x\uparrow \Rightarrow z\uparrow) \Big\} \text{ given } x=1; y=1 \quad \frac{\partial z}{\partial x} = \frac{2(1)}{((1)+(1))^2} = \frac{1}{2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)\frac{\partial (x-y)}{\partial y} - (x-y)\frac{\partial (x+y)}{\partial y}}{(x+y)^2}$$

$$= \frac{(x+y)(-1) - (x-y)(1)}{(x+y)^2} = \frac{-2x}{(x+y)^2} < 0 \ (y\uparrow \Rightarrow z\downarrow) \Big\} \text{ given } x=1; y=1 \quad \frac{\partial z}{\partial y} = \frac{-2(1)}{((1)+(1))^2} = -\frac{1}{2}$$

$$dz = \frac{2y}{(x+y)^2}\cdot dx + \frac{(-2x)}{(x+y)^2}\cdot dy$$

$$= \frac{1}{2}\cdot dx - \frac{1}{2}\cdot dy$$

Therefore, when $x \uparrow$ by 2 unit, z will $\uparrow$ by $\frac{1}{2} \times 2 = 1$ units

when $y \downarrow$ by 2 unit, z will $\uparrow$ by $-\frac{1}{2} \times (-2) = 1$ units

0.3)

$$\frac{\partial z}{\partial x} = 4xy + 3y = 4(1)(2) + 3(2) = 8 + 6 = 14$$

$$\frac{\partial z}{\partial y} = 2x^2 + 3x + 2y = 2(1)^2 + 3(1) + 2(2) = 2+3+4 = 9$$

$$\frac{\partial x}{\partial s} = 2r = 2(1) = 2$$

$$\frac{\partial x}{\partial r} = 2r + 2s = 2(1) + 2(0) = 2$$

$$\frac{\partial y}{\partial s} = -4$$

$$\frac{\partial y}{\partial r} = 2$$

given $r=1, s=0$

$x = r^2 + 2rs = (1)^2 + 2(1)(0) = 1$

$y = 2r - 4s = 2(1) - 4(0) = 2$

$$dz = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial s}ds\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial s}ds\right)$$

- find $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$

$$= (4xy + 3y)(2r) + (2x^2 + 3x + 2y)(-4)$$
$$= 14(2) + 9(-4) = 28 - 36 = -8$$

- find $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r}\right)$

$$= (4xy + 3y)(2r + 2s) + (2x^2 + 3x + 2y)(2)$$
$$= 14(2) + 9(2) = 28 + 18 = 46$$

0.4)

$$2x^2 + 3y^2 + 2z^2 = 16$$

$$F(x,y,z) = 2x^2 + 3y^2 + 2z^2 - 16 = 0$$

$$\frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{6y}{4z} = -\frac{3y}{2z}$$

given $x=1, y=2, z=-1$

$$\frac{dz}{dy} = \frac{-3(2)}{2(-1)} = \frac{-6}{-2} = 3$$

0.5)

$\ln(x+y+z) + xyz = ze^{x+y+z}$ given $x=0,\ y=1,\ z=0$

$f(x,y,z) = \ln(x+y+z) + xyz - ze^{x+y+z}$

$$\frac{dz}{dx} = -\frac{F_x}{F_z}$$

$$= -\frac{1}{1-e}$$

$$F_x = \frac{\partial F}{\partial x} = \frac{1}{(x+y+z)}(1) + yz - ze^{x+y+z}$$

$$= \frac{1}{(\underset{0}{x}+\underset{1}{y}+\underset{0}{z})} + \underset{0}{yz} - \underset{0}{z}e^{x+y+z} = 1$$

$$F_z = \frac{\partial F}{\partial z} = \frac{1}{(x+y+z)}(1) + xy - \left(z\frac{d(e^{x+y+z})}{dz} + e^{x+y+z}\frac{d(z)}{dz}\right)$$

$$= \frac{1}{(x+y+z)} + \underset{0}{xy} - \underset{0}{z}e^{x+y+z} - \underset{1}{e^{x+y+z}} = 1-e$$

Question 1: Suppose that the demand of a product is given by,

$$Q_x = 100 - 4P_x - \frac{50}{\sqrt{P_y}} + 0.5I^2.$$

where P_x is the price of good X, P_y is the price of good Y, and I is the level of income.

Consider the following problems

1.1) What is the relationship between good x and good y? Are they substitute product/complementary product? Show your results using the partial derivative

1.2) Is the product X considered an inferior product?

1.3) What is the level of quantity demanded if $P_x = 10, P_y = 25$ and $I = 10$?

1.4) Calculate the own-price elasticity of demand, and evaluate the value when $P_x = 10, P_y = 25$ and $I = 10$.

1.5) Calculate the cross-price elasticity of demand when $P_x = 10, P_y = 25$ and $I = 10$.

1.6) Calculate income elasticity of demand when $P_x = 10, P_y = 25$ and $I = 10$. Is the product a necessary or luxurious product?

1.1) $Q_x = f(P_x, P_y, I)$

- Partial elasticity of Q_x with respect on P_y

$$= \frac{\partial Q_x}{\partial P_y} \times \frac{P_y}{Q_x} = \frac{\partial \ln(Q_x)}{\partial \ln(P_y)}$$

$$Q_x = 100 - 4P_x - \frac{50}{\sqrt{P_y}} + 0.5I^2$$

$$\frac{\partial Q_x}{\partial P_y} = -\left(\frac{-1}{2}\right) 50 P_y^{\frac{-1}{2}-1}$$

$$= 25 P_y^{\frac{-3}{2}} = \frac{25}{P_y^{3/2}} > 0$$

$P_y \uparrow \rightarrow Q_x \uparrow$

$\therefore$ good x and good y are substitute product

1.2) $\frac{\partial Q_x}{\partial I} = (0.5)(2)I$

$= I > 0$

$I \uparrow \rightarrow Q_x \uparrow$

$\therefore$ Product x doesn't considered inferior product

1.3) $Q_x = 100 - 4P_x - \frac{50}{\sqrt{P_y}} + 0.5\, I^2$

given $P_x = 10, P_y = 25$, and $I = 10$

$$Q_x = 100 - 4(10) - \frac{50}{\sqrt{25}} + 0.5(10)^2$$

$$= 100 - 40 - \frac{50}{5} + 50$$

$$= 85$$

1.4) $\varepsilon_{Q_x, P_x} = \frac{\partial Q_x}{\partial P_x} \cdot \frac{P_x}{Q_x}$

$$= -4 \cdot \frac{10}{100 - 4(10) - \frac{50}{\sqrt{25}} + 0.5(10)^2}$$

$$= -4 \cdot \frac{10}{85}$$

$$= -\frac{40}{85} = -0.47$$

1.5) $\varepsilon_{Q_x, P_y} = \frac{\partial Q_x}{\partial P_y} \cdot \frac{P_y}{Q_x}$

$$= \frac{25}{P_y^{3/2}} \cdot \frac{25}{85}$$

$$= \frac{\cancel{25}}{\cancel{25}\cancel{\sqrt{25}}} \cdot \frac{\overset{5}{\cancel{25}}}{85}$$

$$= 0.0588$$

1.6) $\varepsilon_{Q_x, I} = \frac{\partial Q_x}{\partial I} \cdot \frac{I}{Q_x}$

$$= I \cdot \frac{10}{85}$$

$$= 10 \cdot \frac{10}{85}$$

$= 1.176 > 1 \Rightarrow$ elastic $\therefore$ the product is luxurious product.

Question 2 Given the production function $Q = f(K, L) = A[K^n + L^n]$ where A is the level of technology, K is capital and L is labor. Suppose that $n > 0$. Consider the following problem.

2.1) To ensure that the above production function exhibits a *decreasing return to scale technology*, what additional restrictions do one need to place on n? $n+n<1$

2.2) Under the assumption used in (2.1), show that the production function satisfies the law of diminishing returns.

2.3) Calculate the marginal rate of technical substitution (MRTS) of labor (L) for capital (K).

2.4) how that MRTS is a decreasing function in L. That is, as labor increases, the value of MRTS decreases.

Suppose that $K(t) = \frac{1}{2}t^2 + 2t + 3$ and $L(t) = e^t + 3$, where $t \geq 0$ is the number of periods from now. Consider the following problem

2.5) Show that Q is increasing over time.

2.6) Compute $\frac{\frac{dQ}{Q}}{dt}$ when t = 0, i.e. growth of output in the initial period.

2.1) $n+n<1$

2.2) $Q = f(K,L) = A[K^n + L^n]$

$dQ = \frac{\partial Q}{\partial K} \cdot dK + \frac{\partial Q}{\partial L} \cdot dL = 0$ | $\frac{\partial Q}{\partial K} \cdot dK = -\frac{\partial Q}{\partial L} \cdot dL$ | $\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = -\frac{dK}{dL}$

$MP_K = \frac{\partial Q}{\partial K} = AnK^{n-1} > 0$ $\quad K\uparrow \rightarrow Q\uparrow \Rightarrow AnK^{n-1}$

$MP_L = \frac{\partial Q}{\partial L} = AnL^{n-1} > 0$ $\quad L\uparrow \rightarrow Q\uparrow \Rightarrow AnL^{n-1}$

$\frac{\partial MP_K}{\partial K} \Rightarrow \frac{\partial^2 Q}{\partial K^2} \Rightarrow An(n-1)K^{(n-1)-1} = An\underbrace{(n-1)}_{\ominus}K^{\overbrace{n-2}^{\ominus}} < 0 \Rightarrow MP_K \downarrow$ as $K\uparrow$

$\frac{\partial MP_L}{\partial L} \Rightarrow \frac{\partial^2 Q}{\partial L^2} \Rightarrow An(n-1)L^{(n-1)-1} = An\underbrace{(n-1)}_{\ominus}L^{\overbrace{n-2}^{\ominus}} < 0 \Rightarrow MP_L \downarrow$ as $L\uparrow$

Law of Diminishing Marginal Product
Output ↑ at "diminishing" rate

given $n+n<1$, meaning $n<1$ so $n-1<0$

2.3) $MRTS_{LK} = \frac{MP_L}{MP_K} = -\frac{dK}{dL}$

$= \frac{\cancel{An}L^{n-1}}{\cancel{An}K^{n-1}}$

$= \frac{L^{n-1}}{K^{n-1}}$

$n-1<0 \therefore MRTS_{LK} = \frac{K^{n-1}}{L^{n-1}}$

2.4) From 2.3) $MRTS_{LK} = \frac{K^{n-1}}{L^{n-1}}$

Therefore, when $L \uparrow$, $MRTS_{LK}$ will $\downarrow$

2.5) $dQ = \frac{\partial Q}{\partial K} \cdot \frac{\partial K}{\partial t} \, dt + \frac{\partial Q}{\partial L} \cdot \frac{\partial L}{\partial t} \, dt = 0$

$$\frac{\partial K}{\partial t} = \frac{1}{2}(2)t + 2 = t+2$$

$$\frac{\partial L}{\partial t} = e^t$$

$$\therefore dQ = \underbrace{AnK^{n-1}}_{L\oplus}(t+2)\, dt + \underbrace{AnL^{n-1}}_{L\oplus} e^t \, dt > 0$$

Therefore, Q is increasing over time

2.6)

compute $\frac{\frac{dQ}{Q}}{dt}$ when $t=0$

$$\% \Delta Q \approx \frac{dQ}{Q} = \frac{1}{Q} dQ$$
$$= d\ln Q$$

$Q = A[K^n + L^n]$
$= A(K+L)^n$

$\ln Q = \ln A + n \ln (K+L)$
$= \ln A + n \ln\left[\left(\frac{1}{2}t^2+2t+3\right) + (e^t+3)\right]$

$= \ln A + n \ln\left(\frac{1}{2}t^2+2t+6+e^t\right)$

$$\frac{\partial \ln Q}{\partial t} = \frac{n}{\left(\frac{1}{2}t^2+2t+6+e^t\right)} \cdot \frac{dt\left(\frac{1}{2}t^2+2t+6+e^t\right)}{dt}$$

$$= \frac{n(t+2+e^t)}{\left(\frac{1}{2}t^2+2t+6+e^t\right)}$$

$$t=0; \quad \frac{\frac{dQ}{Q}}{dt} = \frac{n(0+2+e^0)}{\frac{1}{2}(0)^2+2(0)+6+e^0} = \frac{n(2+1)}{6+1} = \frac{3n}{7}$$

Question 3: Suppose that the preference set of a household can be given by

$$U(x,y) = x^{1/2} + y^{1/2},$$

where x is the amount of consumption on good-x, and y is the amount of consumption on good-y. Consider the following problems.

3.1) Calculate the marginal utility of good x and good y, respectively.

3.2) Does the utility function satisfy with the law of diminishing marginal utility?

3.3) Does the marginal utility curve of good x shift up when the consumer consumes more units of good-y?

3.4) What is the level of the household utility when the consumer consumes 1 unit of good-x and 2 units of good-y?

3.5) Following (3.4), use the total differential to calculate the change in the level of utility under which the consumer increases the consumption on good x by 3 units and reduces the consumption on good y by 1 unit.

3.6) Derive the MRS and show that MRS is decreasing in x.

3.1) $MU_x = \frac{\partial U}{\partial x} = \frac{1}{2} x^{-\frac{1}{2}}$

$MU_y = \frac{\partial U}{\partial y} = \frac{1}{2} y^{-\frac{1}{2}}$

3.2) $dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$

$\frac{\partial MU_x}{\partial x} = \frac{\partial^2 U}{\partial x^2} = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1}$

$= -\frac{1}{4} x^{-\frac{3}{2}} < 0 \Rightarrow x \uparrow \rightarrow MU_x \downarrow$

$\frac{\partial MU_y}{\partial y} = \frac{\partial^2 U}{\partial y^2} = \frac{1}{2}\left(-\frac{1}{2}\right) y^{-\frac{1}{2}-1}$

$= -\frac{1}{4} y^{-\frac{3}{2}} < 0 \Rightarrow y \uparrow \rightarrow MU_y \downarrow$

Law of diminishing marginal utility

3.3) $\frac{\partial MU_x}{\partial y} = 0$

$\therefore$ When consumer consumes more units of good-y, marginal utility curve of good-x doesn't change

3.4) $U(x,y) = x^{1/2} + y^{1/2}$

$= \sqrt{1} + \sqrt{2} = 1 + \sqrt{2}$

3.5) $dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$

$= \frac{1}{2} x^{-\frac{1}{2}} (3) + \frac{1}{2} y^{-\frac{1}{2}} (-1)$

$= \frac{3}{2\sqrt{x}} - \frac{1}{2\sqrt{y}}$

$= \frac{3}{2\sqrt{1}} - \frac{1}{2\sqrt{2}} = 1.146$

3.6) $MRS = \frac{MUx}{MUy} = \frac{dy}{dx}$

$= \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}$

$= \frac{\sqrt{y}}{\sqrt{x}}$

$\frac{\partial MRS}{\partial x} = -\frac{1}{2} x^{-\frac{1}{2}-1} y^{\frac{1}{2}}$

$= \frac{-1}{2} x^{-\frac{3}{2}} y^{\frac{1}{2}}$

$= \frac{-\sqrt{y}}{2x\sqrt{x}} < 0$

$\therefore$ MRS is decreasing in x

Question 4: Suppose that a firm produces $Q = f(L) = L^{1/2}$ units of commodity using *L* units of labor. If the firm gets *P* baht per unit produced and pays *w* baht for a unit of labor, write down the profit function, and find the first-order condition for profit maximization at $L^* > 0$. Solve for L^* and calculate $\frac{\partial L^*}{\partial w}$ *and* $\frac{\partial L^*}{\partial P}$.

4.)

$$\pi = TR - TC$$
$$= PQ - WL$$
$$= P(L^{1/2}) - WL$$

FOC: $\frac{\partial \pi}{\partial L} = \frac{1}{2} P L^{-\frac{1}{2}} - W = 0$

$$\frac{P}{2\sqrt{L}} = W$$

$$\frac{P}{2W} = \sqrt{L}$$

$$L = \left(\frac{P}{2W}\right)^2$$

$$L^* = \frac{P^2}{4W^2} = \frac{1}{4} P^2 W^{-2}$$

$$\frac{\partial L^*}{\partial W} = \frac{1}{4}(-2) P^2 W^{-3}$$
$$= -\frac{P^2}{2W^3} \quad < 0 \; ; \; W \uparrow \rightarrow L^* \downarrow$$

$$\frac{\partial L^*}{\partial P} = \frac{1}{4}(2) P W^{-2}$$
$$= \frac{P}{2W^2} \quad > 0 \; ; \; P \uparrow \rightarrow L^* \uparrow$$