EE320 (2/2012)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION WITHOUT CONSTRAINTS:

MORE-THAN-ONE-INDEPENDENT VARIABLE CASES

Topics

- The differential version of optimization conditions
- Two choice variable optimization
  - Conditions for maximum or minimum
  - Economics examples:
    - Multiplant-firm
    - Multiproduct-firm
    - Third degree price discrimination
- Multivariable optimization
  - Conditions for maximum or minimum
  - Economics examples

Differential Version of Optimization Conditions

- Given $z = f(x)$, the differential of z is:

  $dz = f'(x)dx$, where $f'(x) = dz/dx$.

- The first-order condition (in terms of derivative) for an optimum:

➢ The first-order differential condition for any arbitrary $dx = 0$:

  $dz = 0$.

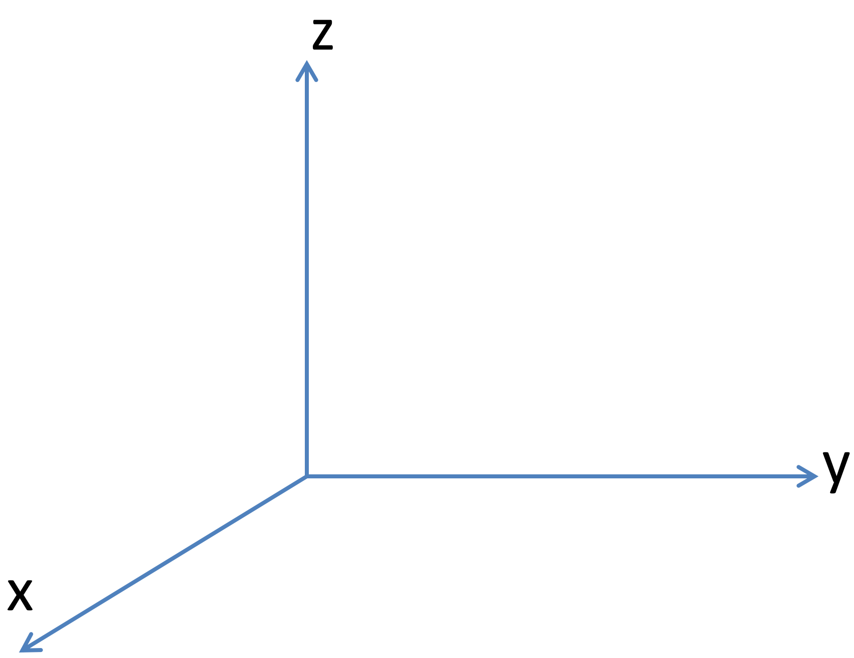
Differential Version of Optimization Conditions

- The second-order sufficient condition (in terms of derivative):
  - For a maximum of z:
  - For a minimum of z:
- Rewrite the S.O.C. in terms of differential:
- The second-order differential condition for any arbitrary nonzero value of z:
  - For a maximum of z:
  - For a minimum of z:

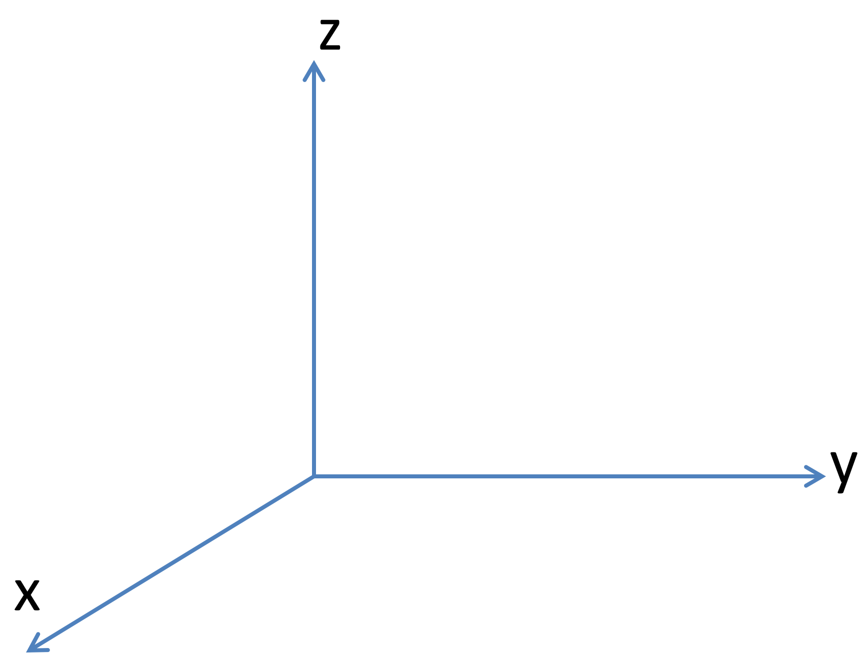
TWO CHOICE VARIABLE OPTIMIZATION

Extreme Values of a Function of Two Variables: $z = f(x, y)$

Maximum value



Minimum value



First-Order Condition

- For *z = f(x, y),* the first-order necessary condition is:

  dz = 0 for arbitrary values of *dx* and *dy*, not both zero.

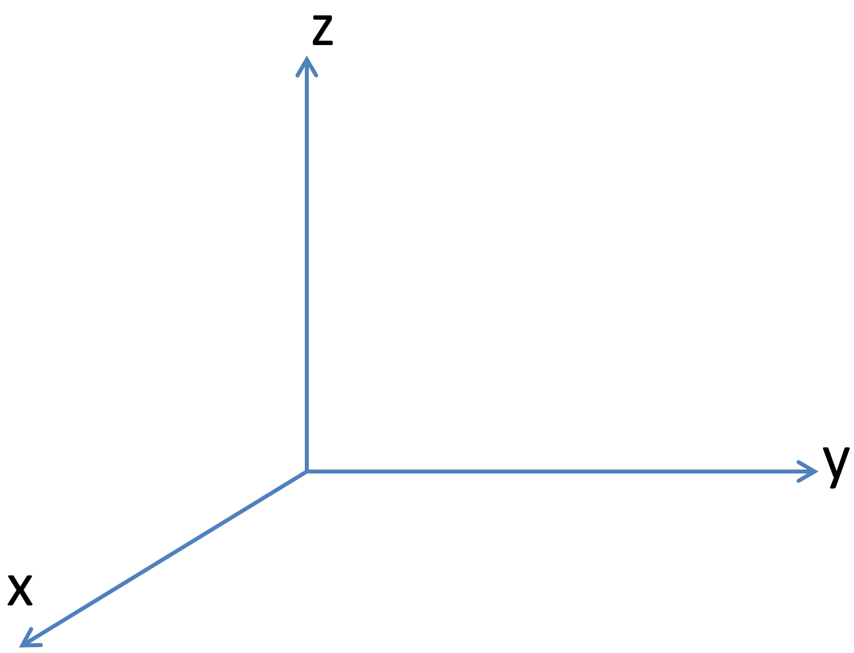
- The total differential is:

Theorem: A differentiable function *z = f(x, y)* can only have a maximum or minimum at an interior point (x_0, y_0) if it is a stationary point. That is, if the point $(x, y) = (x_0, y_0)$ satisfies the two F.O.C. equations:

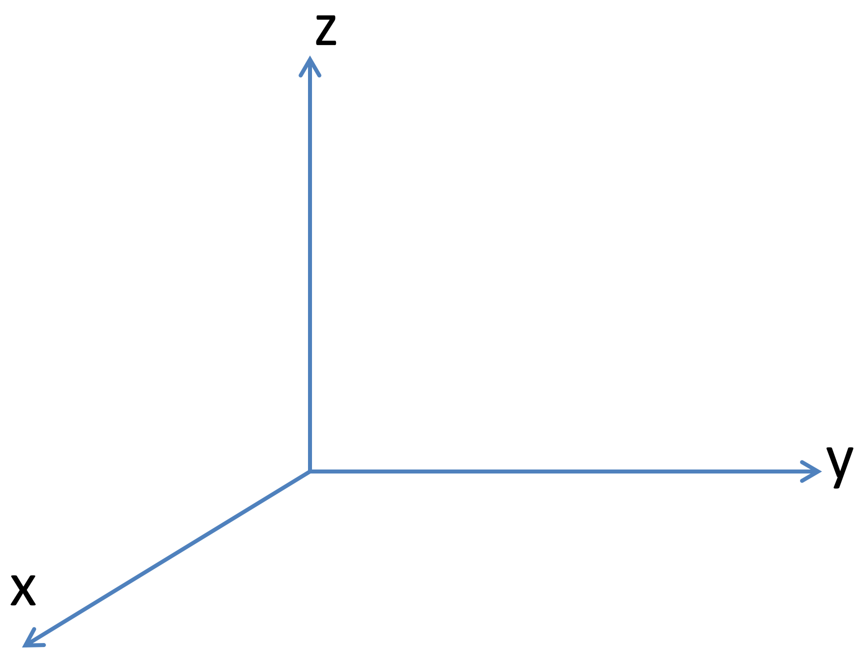
$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0.$$

The FOC is necessary but *not sufficient* condition.

Saddle Point



Inflection Point



Second-Order Condition

- Second-order partial derivatives:
  - f_{xx} = ; f_{xy} =
  - f_{yy} = ; f_{yx} =
- Second-order total differentials:

Second-Order Condition

- **Example**: Let $z = x^3 + 5xy - y^2$. Find dz and d^2z. Then, evaluate dz and d^2z at $x = 1$ and $y = 2$.

Second-Order Condition

- Given that the first-order necessary condition is satisfied, the second-order sufficient condition for z = f(x, y) is:
  - For maximum, $d^2z < 0$ for arbitrary values of dx and dy, not both zero.
  - For minimum, $d^2z > 0$ for arbitrary values of dx and dy, not both zero.

- Equivalent conditions on second-order partial derivatives:

  1. $d^2z < 0$ iff $f_{xx} < 0$; $f_{yy} < 0$; and $f_{xx} f_{yy} > (f_{xy})^2$.
  2. $d^2z > 0$ iff $f_{xx} > 0$; $f_{yy} > 0$; and $f_{xx} f_{yy} > (f_{xy})^2$.

Note: The second-order partial derivatives are to be evaluated at the stationary point where $f_x = f_y = 0$.

Necessary and Sufficient Conditions for a Maximum and a Minimum

Condition	Maximum	Minimum
First-order necessary	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient	$f_{xx}, f_{yy} < 0$ *and* $f_{xx}f_{yy} > (f_{xy})^2$	$f_{xx}, f_{yy} > 0$ *and* $f_{xx}f_{yy} > (f_{xy})^2$

Examples: 2-choice variable optimization

- **Example1**: $z = f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158.$

Second-Order Total Differential as a Quadratic Form

- Consider *dx* and *dy* as variables and partial derivatives as coefficients; i.e.,

  $u = dx$; $v = dy$

  $a = f_{xx}$; $b = f_{yy}$; $h = f_{xy}$ $[=f_{yx}]$

- From $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$, and let $q = d^2z$.

Rewrite d^2z as:

- <u>Question</u>: What restriction must be placed on *a, b*, and *h*, in order to determine a *definite* sign of *q*?

Positive and Negative Definiteness

- A quadratic form is said to be

positive definite	if q is invariably	*positive (> 0)*
positive semidefinite		*non-negative (≥ 0)*
negative semidefinite		*non-positive (≤ 0)*
negative definite		*negative (< 0)*

- If q changes signs when the variables (dx, dy) take different values, then q is said to be *indefinite* (e.g. saddle point).

Determinantal Test for Sign Definiteness

- From $q = au^2 + 2huv + bv^2$

➢ q is $\begin{bmatrix} \text{positive definite} \\ \text{negative definite} \end{bmatrix}$ iff $\begin{bmatrix} \quad \\ \quad \end{bmatrix}$ and $ab-h^2 > 0$.

Determinantal Test for Sign Definiteness

- Rewrite $q = au^2 + 2huv + bv^2$ in the matrix form:

➢ *q* is $\left\{ \begin{array}{l} \textit{positive definite} \\ \textit{negative definite} \end{array} \right\}$ iff $\left\{ \begin{array}{l} \\ \\ \end{array} \right\}$ and .

where

|a| = a is the first leading principal minor of D;

$\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ is the second leading principal minor of D.

Determinantal Test for Sign Definiteness

- Thus, the second-order total differential

  1. d^2z is *positive definite* iff $f_{xx} > 0$ & $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2 > 0$

     ➔ z is a .

  2. d^2z is *negative definite* iff $f_{xx} < 0$ & $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2 > 0$

     ➔ z is a .

- In the matrix form $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$,

  The determinant with the second-order partial derivatives as its elements is called a "Hessian determinant":

Example: Multiproduct Firm (1)

- Competitive firm

  Let $p_1 = 6$, $p_2 = 9$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

  Find Q_1 and Q_2 that maximize the firm's profit.

Example: Multiproduct Firm (2)

- Monopolist

  Let $p_1 = 35 - Q_1$, $p_2 = 33 - Q_2$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

  Find Q_1 and Q_2 that maximize the firm's profit.

Example: Multiplant Firm

- Let P = 25, $TC_1 = 2Q_1^2 + 5Q_1 + 10$, and $TC_2 = 2Q_2^2 + 3Q_2 + 15$.
  Find Q_1 and Q_2 that maximize the firm's profit.

Example:
Multimarket Monopoly (Price Discrimination)

- Let $R = R_1(Q_1) + R_2(Q_2)$ and $C = C(Q)$ where $Q = Q_1 + Q_2$. Find the FONC and SOSC for profit maximization.

Example: Price Discrimination

- Let $p_1 = 22 - 2Q_1$, $p_2 = 10 – 0.5Q_2$ and TC = 2Q + 5.
  Find Q_1, Q_2, p_1, and p_2 that maximize the firm's profit.

Example: Input Decision of a firm

- Let $Q = f(K, L) = 5K^{0.5}L^{0.25}$, $P = 4$, $w = 10$, $r = 5$. Find K^* and L^* that maximize profit.

MULTIVARIABLE OPTIMIZATION

Objective Functions with 3 Choice Variables

- Let $z = f(x_1, x_2, x_3)$. ➔ $dz =$

➢ First-order necessary condition for an extremum of z is:

- Second-order total differential:

  $d^2z =$

➔Hessian matrix:

Objective Functions with 3 Choice Variables

- The leading principal minors of the Hessian matrix are:

- Thus, second-order sufficient condition for an extremum of z:

1. z^* is a *minimum* if $|H_1| > 0$; $|H_2| > 0$; $|H_3| > 0$
   (i.e. d^2z is *positive definite*.)

2. z^* is a *maximum* if $|H_1| < 0$; $|H_2| > 0$; $|H_3| < 0$
   (i.e. d^2z is *negative definite*.)

Note: All the leading principal minors are evaluated at the stationary points where $f_1 = f_2 = f_3 = 0$.

n-variable Optimization

- Let $z = f(x_1, x_2, \ldots, x_n)$.
- Total differential:

➢ First-order necessary condition:

➢ Second-order sufficient condition:

Consider the Hessian matrix:

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

where $|H_1|, |H_2|, \ldots, |H_n|$ are the leading principal minors.

For a *miminum* in z, all the principal minors must be *positive*.

For a *maximum* in z, all the principal minors *alternate is sign, the first one being negative*.

Necessary and Sufficient Conditions for a Maximum and Minimum (*n*- choice variables)

Condition	Maximum	Minimum
First-order necessary	$f_1 = f_2 = \ldots = f_n = 0$	$f_1 = f_2 = \ldots = f_n = 0$
Second-order sufficient	$\lvert H_1 \rvert < 0$, $\lvert H_2 \rvert > 0$, $\lvert H_3 \rvert < 0$, … or $(-1)^i \lvert H_i \rvert > 0$ for i = 1, 2, …, n	$\lvert H_1 \rvert > 0$, $\lvert H_2 \rvert > 0$, $\lvert H_3 \rvert > 0$, … Or $\lvert H_i \rvert > 0$ for i = 1, 2, …, n

Example: Multimarket Monopoly

- Let $p_1 = 63 - 4Q_1$, $p_2 = 105 – 5Q_2$, $p_3 = 75 – 6Q_2$ and $TC = 15Q + 20$. Find Q_i and p_i , for i = 1, 2,3, that maximize the firm's profit.