SUBJECT: EE320 - Implicit Function NO: 1. DATE: / /

Assume $F(y, x_1, x_2, \ldots, x_n) = 0$ implicitly defines $y = f(x_1, x_2, \ldots, x_n)$.

Implicit Function Rule : $\boxed{\dfrac{\partial y}{\partial x_i} = -\dfrac{F_i}{F_y}}$

where $F_i \equiv \dfrac{\partial F}{\partial x_i}$ and $F_y \equiv \dfrac{\partial F}{\partial y}$

Proof : $\dfrac{\partial y}{\partial x_i} = \dfrac{-F_i}{F_y}$.

Given $F(y, x_1, x_2, \ldots, x_n) = 0$

Totally differentiate on both sides :

$$dF = \frac{\partial F}{\partial y} . dy + \frac{\partial F}{\partial x_1} . dx_1 + \frac{\partial F}{\partial x_2} . dx_2 + \ldots + \frac{\partial F}{\partial x_n} . dx_n = d(0) = 0.$$

$$F_y\, dy + F_1\, dx_1 + F_2\, dx_2 + \ldots + F_n\, dx_n = 0 \quad \text{—— (1)}$$

From $y = f(x_1, x_2, \ldots, x_n)$, totally differentiation gives :

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

$$dy = f_1\, dx_1 + f_2\, dx_2 + \ldots + f_n\, dx_n \quad \text{—— (2)}$$

Substitute dy from (2) in (1) :

$$F_y \overbrace{(f_1\, dx_1 + f_2\, dx_2 + \ldots + f_n\, dx_n)}^{dy} + F_1\, dx_1 + F_2\, dx_2 + \ldots + F_n\, dx_n = 0.$$

$$\Rightarrow (F_y f_1 + F_1)\, dx_1 + (F_y f_2 + F_2)\, dx_2 + \ldots + (F_y f_n + F_n)\, dx_n = 0. \quad \text{—— (3)}$$

Since $dx_i \neq 0$ for $i = 1, \ldots, n$, $F_y f_i + F_i$ must be zero for (3) to be true.

$$\Rightarrow F_y f_i + F_i = 0 \quad \Rightarrow \boxed{f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}}$$

Q.E.D.

Application : Partial Market Equilibrium.

Let $Q_d = D(P, Y_0)$ where $\partial D/\partial P < 0$, $\partial D/\partial Y_0 > 0$

$Q_s = S(P)$ where $\partial S/\partial P > 0$

At eqm, $Q_d = Q_s \Rightarrow D(P, Y_0) = S(P)$

$\Rightarrow D(P, Y_0) - S(P) = 0.$

Let P^* be the equilibrium price where $P^* = P^*(Y_0)$.

$\Rightarrow$ Equilibrium identity : $F(P^*, Y_0) \equiv D(P^*, Y_0) - S(P^*) = 0.$

Questions : $\frac{dP^*}{dY_0} = ?$, $\frac{dQ^*}{dY_0} = ?$

By using implicit function rule, we get:

① $$\frac{dP^*}{dY_0} = -\frac{F_{Y_0}}{F_{P^*}} = -\frac{\partial F/\partial Y_0}{\partial F/\partial P^*}$$

From $F(P^*, Y_0) = D(P^*, Y_0) - S(P^*) = 0$,

$$\frac{\partial F}{\partial Y_0} = \frac{\partial D}{\partial Y_0} > 0$$

$$\frac{\partial F}{\partial P^*} = \underset{\ominus}{\frac{\partial D}{\partial P^*}} - \underset{\oplus}{\frac{dS}{dP^*}} < 0$$

$$\therefore \frac{dP^*}{dY_0} = -\frac{F_{Y_0}}{F_{P^*}} = -\frac{\overset{\oplus}{\partial D/\partial Y_0}}{\underset{\ominus}{\frac{\partial D}{\partial P^*}} - \underset{\oplus}{\frac{dS}{dP^*}}} > 0.$$

② $$\frac{dQ^*}{dY_0} = \frac{dS[P^*(Y_0)]}{dY_0} = \underset{\oplus}{\frac{dS}{dP^*}} \cdot \underset{\oplus}{\frac{dP^*}{dY_0}} > 0$$

from above