Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

$$V_c(C_{T-1}, T-1) = E_{T-1}\left[\delta^T \frac{\partial}{\partial W_T}\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right). R_{T-1}\right]$$

$$\delta^{T-1} \cdot C_{T-1}^{-\gamma} = E_{T-1}\left[\delta^T W_T^{-\gamma} R_{T-1}\right]$$

$$= E_{T-1}\left[\delta^T (R_{T-1} S_{T-1})^{-\gamma} R_{T-1}\right]$$

$$= E_{T-1}\left[\delta^T R_{T-1}^{1-\gamma} S_{T-1}^{-\gamma}\right]$$

$$C_{T-1}^{-\gamma} = \delta R_{T-1}^{1-\gamma} S_{T-1}^{-\gamma}$$

$$C_{T-1} = \left[\delta R_{T-1}^{1-\gamma}\right]^{-\frac{1}{2}} S_{T-1}$$

$$= \left[\delta R_{T-1}^{1-\gamma}\right]^{-\frac{1}{2}} (W_{T-1} - C_{T-1})$$

$$= \left[\delta R_{T-1}^{1-\gamma}\right]^{-\frac{1}{2}} W_{T-1} - \left[\delta R_{T-1}^{1-\gamma}\right]^{-\frac{1}{2}} C_{T-1}$$

$$C^*_{T-1} = \frac{\alpha}{1+\alpha} W_{T-1} \quad \text{where } \alpha = \left[\delta R_{T-1}^{1-\gamma}\right]^{-\frac{1}{2}}$$

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score......................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$W_T = S_{T-1}R_{T-1}$

$$J(W_{T-1},T-1) = \max_{c,w_i} U_{T-1} + E_{T-1}[J(W_T,T)]$$

$$= \delta^{T-1}\frac{C_{T-1}^{*(1-\gamma)}}{1-\gamma} + \frac{\delta^T}{1-\gamma}\left[R_{T-1}^*(W_{T-1}-C_{T-1}^*)\right]^{1-\gamma}$$

note from (1.1) $C_{T-1}^* = \frac{\alpha}{1+\alpha}W_{T-1}$ where $\alpha = \left[\delta R_{T-1}^{1-\gamma}\right]^{-\frac{1}{\gamma}}$

$$= \delta^{T-1}\frac{\left(\frac{\alpha}{1+\alpha}W_{T-1}\right)^{1-\gamma}}{1-\gamma} + \frac{\delta^T}{1-\gamma}\left[R_{T-1}^*\left(W_{T-1}-\frac{\alpha}{1+\alpha}W_{T-1}\right)\right]^{1-\gamma}$$

$$= \frac{\delta^T}{1-\gamma}\left[\delta^{-1}\left(\frac{\alpha}{1+\alpha}W_{T-1}\right)^{1-\gamma} + \left[R_{T-1}^*W_{T-1}\left(1-\frac{\alpha}{1+\alpha}\right)\right]^{1-\gamma}\right]$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

using condition for optimal consumption,

$$V_c(C^*_{T-2}) = E_{T-2}\left[J_W(W_{T-1}, T-1)\, R_{T-2}\right]$$

note from (1.2), $J_{T-1} = \frac{\delta^T}{1-\gamma}\left[\delta^{-1}\left(\frac{\alpha}{1+\alpha}W_{T-1}\right)^{1-\gamma} + \left[R^*_{T-1}W_{T-1}\left(1-\frac{\alpha}{1+\alpha}\right)\right]^{1-\gamma}\right]$

$$\delta^{T-2}C_{T-2}^{-\gamma} = E_{T-2}\left[\delta^T\left(\delta^{-1}\left(\frac{\alpha}{1+\alpha}\right)^{1-\gamma}W_{T-1}^{-\gamma} + R^*_{T-1}\left(1-\frac{\alpha}{1+\alpha}\right)\right)\right]$$

$$C^*_{T-2} = \left(\delta^2\left[\delta^{-1}\left(\frac{\alpha}{1+\alpha}\right)^{1-\gamma}W_{T-1}^{-\gamma} + R^*_{T-1}\left(1-\frac{\alpha}{1+\alpha}\right)\right]\right)^{-\frac{1}{2}}$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = \max_{c, w_2} V_{T-2} + E_{T-2}\left[J(W_{T-1}, T-1)\right]$$

$$= \delta^{T-2} \frac{c_{T-2}^{*(1-\gamma)}}{1-\gamma} + E_{T-2}\left[\frac{\delta^{T}}{1-\gamma} \left[\delta^{-1} \left(\frac{\alpha}{1+\alpha} W_{T-1}\right)^{1-\gamma} + \left[R^*_{T-1} W_{T-1} \left(1-\frac{\alpha}{1+\alpha}\right)\right]^{1-\gamma} \right]\right]$$

$$= \frac{\delta^{T-2}\left(\delta^{2}\left[\delta^{-1}\left(\frac{\alpha}{1+\alpha}\right)^{1-\gamma} W_{T-1}^{-\gamma} + R^*_{T-1}\left(1-\frac{\alpha}{1+\alpha}\right)\right]\right)^{-\frac{1-\gamma}{2}}}{1-\gamma}$$

$$+ E_{T-2}\left[\frac{\delta^{T}}{1-\gamma} \left[\delta^{-1} \left(\frac{\alpha}{1+\alpha} W_{T-1}\right)^{1-\gamma} + \left[R^*_{T-1} W_{T-1} \left(1-\frac{\alpha}{1+\alpha}\right)\right]^{1-\gamma} \right]\right]$$