Chapter 2

Statistics revision

Flow of study in this chapter

› Probability, random variable and density

Revision the basic concept of probability and its distribution, graphing random variable and its probability.

› Bivariate probability density

When two random variables, or two events, coexist, what aspects of the distribution can be studied.

› Central tendency and dispersion

How to measure, and why, central tendency and dispersion of a distribution for a random variable.

› Common distribution functions

The distributions we are relying on for statistical in this semester.

Further reading can be found in Gujarati and Porter (2009), Appendix A, page 801-837

(1) Event, Sample Space and Probability

Let A be an event of interest, occurring within a given sample space S and $P(A)$ be the probability that A will occur, $P(A)$ is defined as

› $P(A) = \dfrac{number\ of\ times\ event\ A\ will\ occur}{number\ of\ all\ possible\ outcome\ in\ sample\ space\ S}$

Example tossing 2 fair coins, the sample space is

› $S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$

If the event of interest is having at least a coin turning head (H) is

› $A = \{\text{HH}, \text{HT}, \text{TH}\}$.

The probability of this event is then

› $P(A) =$

(1) Event, Sample Space and Probability

Probability Axioms

(1) $0 \leq P(A) \leq 1$

(2) $P(S) = 1$

(3) If and A are B mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

(2) Random variable

Definition 2.1

Let X be a **random variable,** the results of an experiment in the form of value, which value is given by one of the results.

Example Tossing 2 fair coins again, let X be 0 if the result shows **at least** a coin turned up head, be 1 otherwise. The sample space was defined as

› $S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$

Transforming theses events into random variable, we get

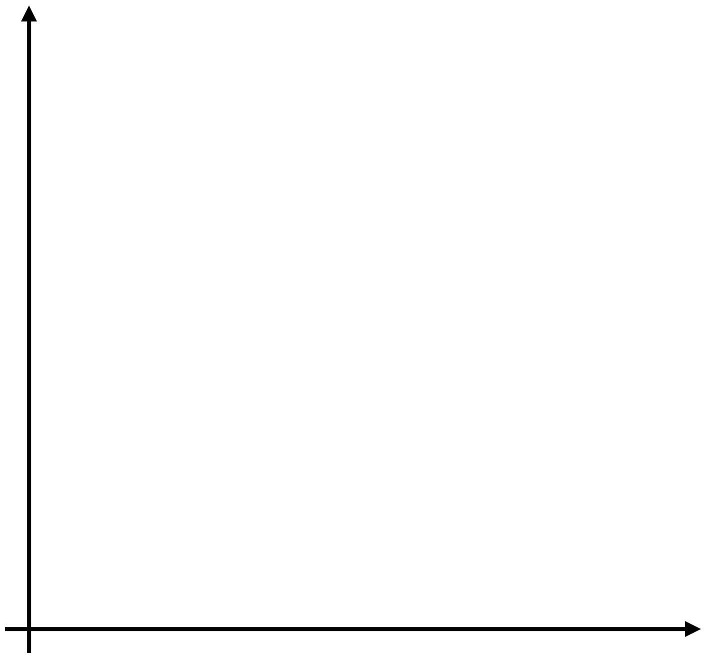
› $S_X = \{0,1\}$

Therefore, if we put probability function with specific value of random variable X, we have

› $P(X = 0) =$

› $P(X = 1) =$

(2) Random variable



Graphing the random variable and its probability here on the left.

› **Discreate random variable** is a random variable that can take specific values of event.

› **Continuous random variable** is a random variable that can takes infinite amounts of value of event.

(3) Probability Density Function (PDF)

Definition 2.2

A function whose value at any given sample (or point) in the sample space (the set of possible values taken by the random variable) can be interpreted as providing a relative likelihood that the value of the random variable would equal that sample.

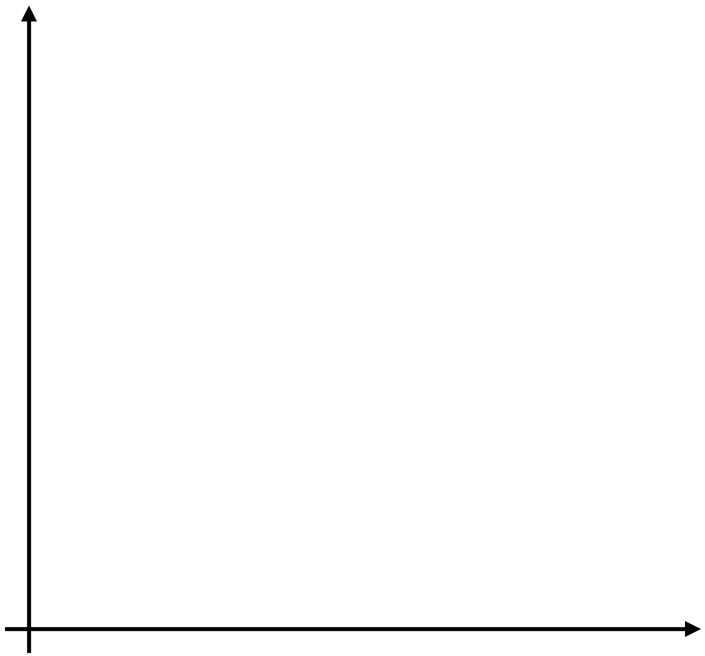
› $f(x_i) = P(X = x_i)$ for $x_i \in S_X$

› $f(x_i) = 0$ for $x_i \notin S_X$

Example Let X be a random variable of total points from rolling 2 fair dices, the sample space would be

› $S_X =$

(3) Probability Density Function (PDF)

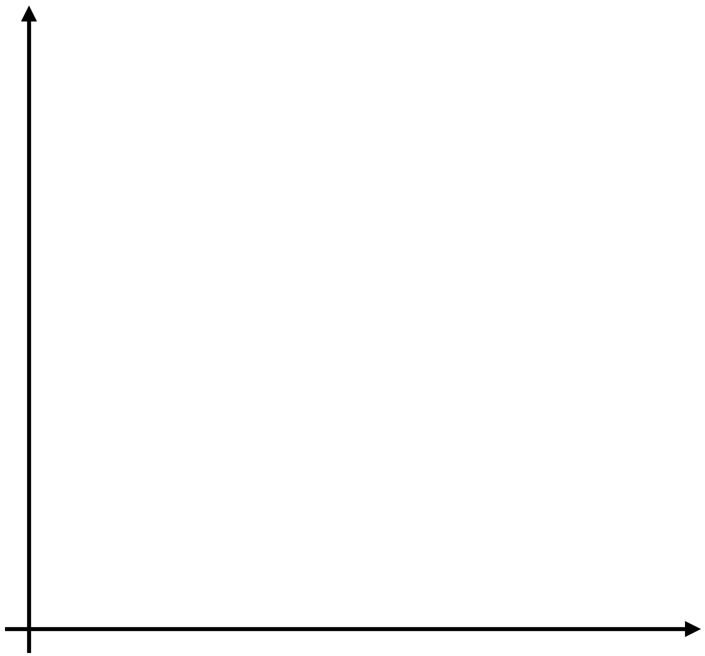


Graphing the random variable and its probability here on the left.

Now figure out these probabilities.

- $P(X = 4) =$
- $P(X = 7) =$
- $P(X < 3) =$
- $P(X \leq 4) + P(X > 9) =$

(3) Probability Density Function (PDF)



PDF can be both discrete and continuous.

Discreate PDF

› $0 \leq f(x) \leq 1$

› $\sum_{-\infty}^{\infty} f(x) = 1$

› $\sum_{a}^{b} f(x) = P(a \leq X \leq b)$

Continuous PDF

› $0 \leq f(x) \leq 1$

› $\int_{-\infty}^{\infty} f(x)dx = 1$

› $\int_{a}^{b} f(x)dx = P(a \leq X \leq b)$

(1) Conditional probability

There are a few basic concepts of bivariate distribution worth revising

› Joint probability density function

› Marginal probability

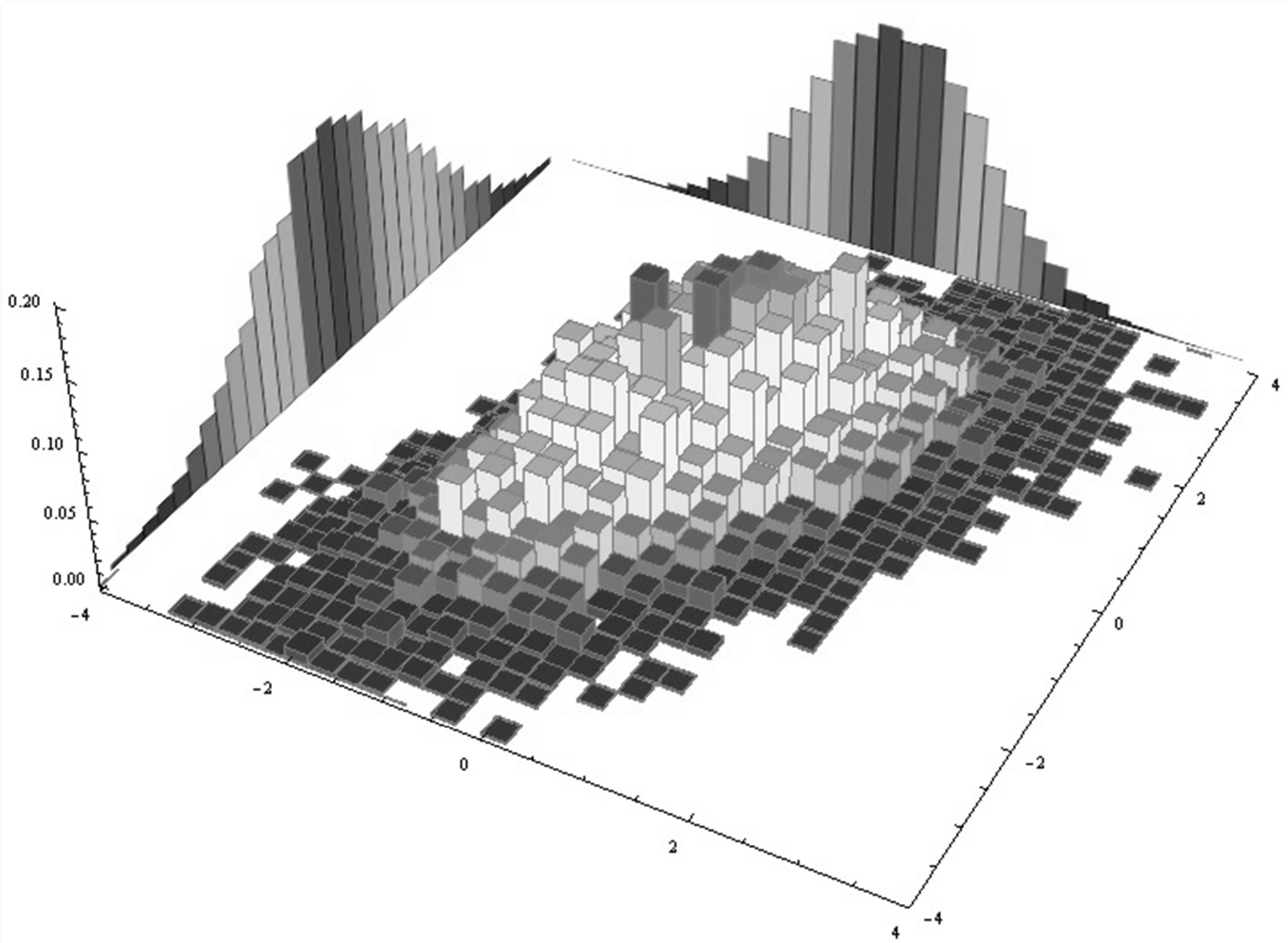
This class we will focus on

Definition 2.3

Conditional probability is a measure of the probability of an event occurring given that another event has (by assumption, presumption, assertion or evidence) occurred. Conditional probability is defined as

› $f(X|Y) = P(X = x|Y = y) = \frac{f(x,y)}{f(y)}$

(1) Conditional probability



(1) Conditional probability

Example Two archers are competing shooting a target for 3 times each.

Let X and Y be number of times archer 1 and archer and archer 2 hit the target respectively.

Thousand of rounds has been competed and the probability is computed as a result in this table.

		X		
		1	2	3
Y	1	0.35	0.2	0.07
	2	0.1	0.05	0.09
	3	0.08	0.04	0.02

› Find $f(X = 1|Y = 2) =$

› Find $f(Y = 2|X = 3) =$

(2) Statistical independence

Definition 2.4

Two random variables are considered **independent** if and only if the condition below is satisfied.

› $f(x, y) = f(x) \cdot f(y)$

Example Using the same archer example, prove that archers' performance is independent.

		X		
		1	2	3
Y	1	1/9	1/9	1/9
	2	1/9	1/9	1/9
	3	1/9	1/9	1/9

›

(1) Expected value

Definition 2.5

Expected value of a random variable is a generalization of the weighted average and intuitively is the arithmetic mean of independent realizations of that variable. Expected value is defined as

- $E(X) = \sum_{i=1}^{n} x_i \cdot f(x_i)$ - discrete
- $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$ - continuous

Grade	Prob
A	0.3
B	0.4
C	0.15
D	0.1
F	0.05

<u>**Example**</u> A student is trying hard for econometrics class. The probability getting grades are listed in the table.

›

(1) Expected value

Properties of expected value

(1) $E(\mathrm{a}) = \mathrm{a}$ for any constant a

(2) $E(\mathrm{a}X) = \mathrm{a}E(X)$

(3) $E(\mathrm{a}X + \mathrm{b}) = \mathrm{a}E(X) + \mathrm{b}$

(4) $E(X \pm Y) = E(X) \pm E(Y)$

(5) $E(XY) = E(X) \cdot E(Y)$
if and only if X and Y are independent.

(1) Expected value

Example Find the expected value of this distribution

$f(X) = \frac{1}{9}x^2$ for $0 \leq x \leq 3$

›

(2) Conditional expectation

Definition 2.6

Let $f(X,Y)$ be a joint probability density function, the expectation of X conditional on some value of Y is

› $E(X|Y) = \sum_X x_i \cdot f(X|Y=y)$ - discrete

› $E(X|Y) = \int_{-\infty}^{\infty} x \cdot f(X|Y=y)dx$ - continuous

		X: -2	0	2	3
Y	3	0.27	0.08	0.16	0
	6	0	0.04	0.1	0.35

Example Find $E(X|Y=3)$ from the PDF given in the table.

›

(3) Variance

Definition 2.7

Variance is a measure of data dispersion from the expected value. Given that μ is the expected value of X, then

› $var(X) = \sum_{i=1}^{n}(x_i - \mu)^2 \cdot f(x_i)$- discrete

› $var(X) = \int_{-\infty}^{\infty}(x - \mu)^2 \cdot f(x)dx$- continuous

Another formula for variance is

› $var(X) = E[(X - \mu)^2] = E(X^2) - \mu^2$

(3) Variance

Properties of variance

(1) $var(\mathrm{a}) = 0$ for any constant a

(2) $var(\mathrm{a}X + b) = \mathrm{a}^2 var(X)$

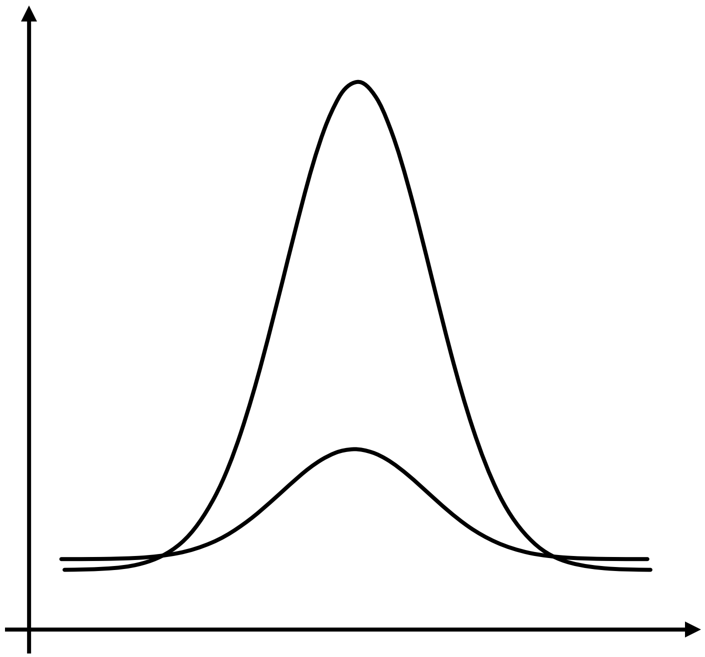
(3) $var(X \pm Y) = var(X) \pm var(Y)$
if and only if X and Y are independent.

X	**-2**	**1**	**2**
$f(X)$	5/8	1/8	2/8

Example Find variance from the PDF given in the table.

›

(3) Variance



Example Find variance from given distribution

› $f(X) = \frac{1}{9}x^2$ for $0 \leq x \leq 3$

(4) Conditional variance

Definition 2.8

Conditional variance is a measure variance, but coupled with a condition on another variable, defined as

› $var(X|Y) = \sum_X [x_i - E(X|Y = y)]^2 \cdot f(X|Y = y)$ - discrete

› $var(X|Y) = \int_{-\infty}^{\infty} [x - E(X|Y = y)]^2 \cdot f(X|Y = y) dx$ - continuous

(5) Covariance

Definition 2.8

Let X and Y be two random variables with expected value of μ_X and μ_Y respectively, the **covariance** is a measure of the joint variability of two random variables.

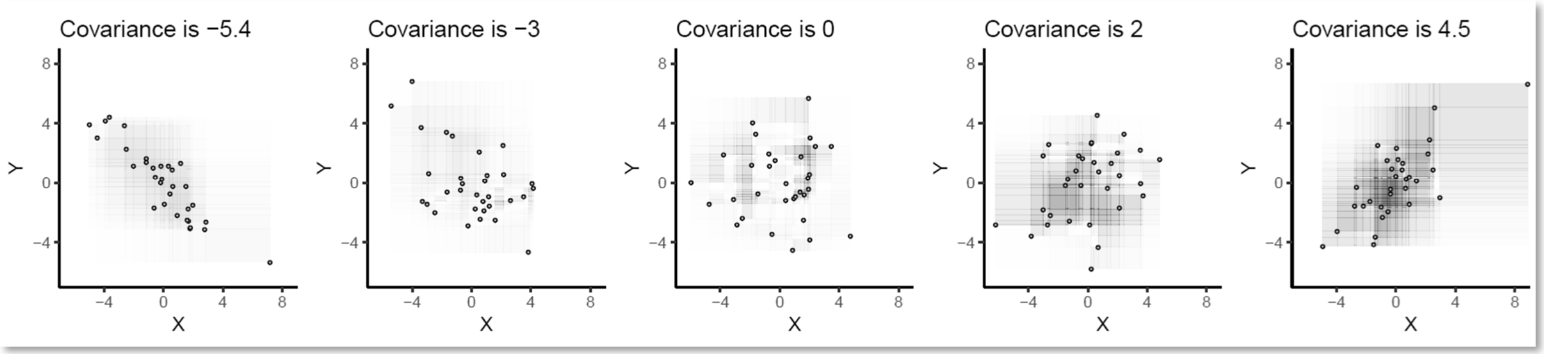
If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the lesser values. Defined as

› $cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\} = E(XY) - \mu_X\mu_Y$ - discrete

› $cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X-\mu_X)(Y-\mu_X)\cdot f(x,y)dxdy - \mu_X\mu_Y$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} XYf(x,y)\,dxdy - \mu_X\mu_Y$ - continuous

(5) Covariance



Further properties of variance

If X and Y are ***not*** independent, then

› $var(X \pm Y) = var(X) + var(Y) \pm 2cov(X,Y)$

Problems with interpretation

"A large covariance can mean a strong relationship between variables. However, you can't compare variances over data sets with different scales (like pounds and inches). A weak covariance in one data set may be a strong one in a different data set with different scales.

(6) Correlation coefficient

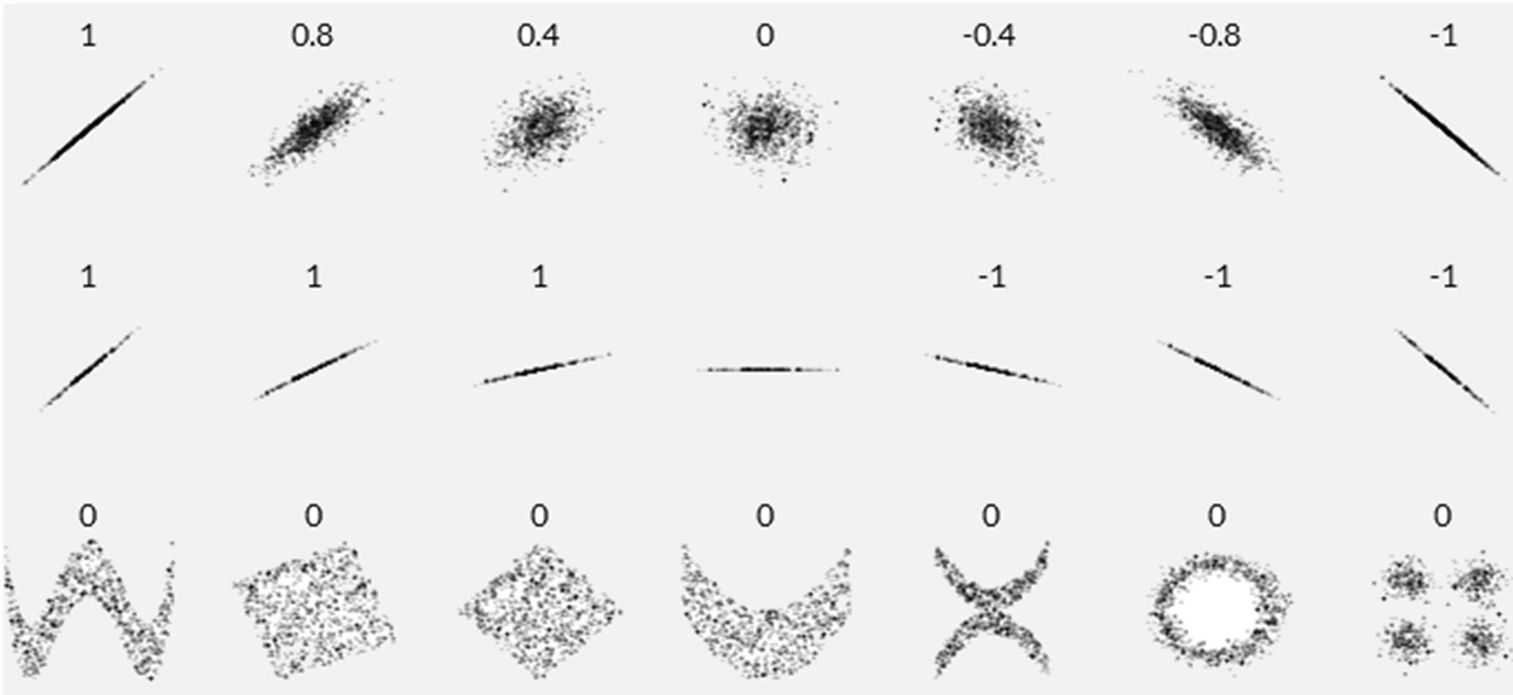
Definition 2.9

Correlation coefficient is a numerical measure of some type of correlation, meaning a statistical relationship between two variables, denoted by ρ_{XY}, r_{XY}, $\mathrm{corr}(X, Y)$.

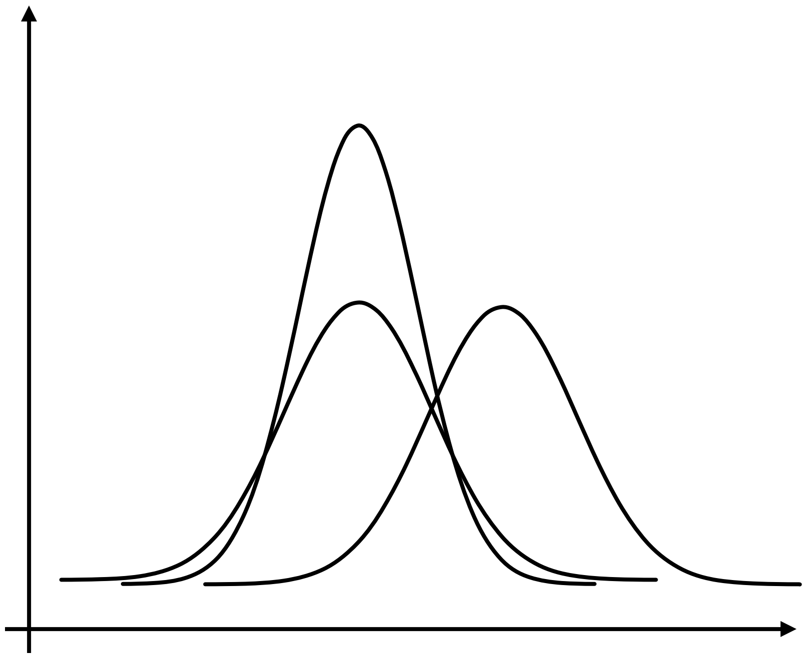
› $\rho_{XY} = \frac{cov(X,Y)}{\sigma_X \sigma_Y} \epsilon [-1,1]$

where σ_X is standard deviation or $\sigma_X = \sqrt{var(X)}$

(6) Correlation coefficient



(1) Normal distribution



A continuous random variable X is normally distributed with mean μ and variance σ^2, denoted as $X \sim N(\mu, \sigma^2)$, if the PDF is

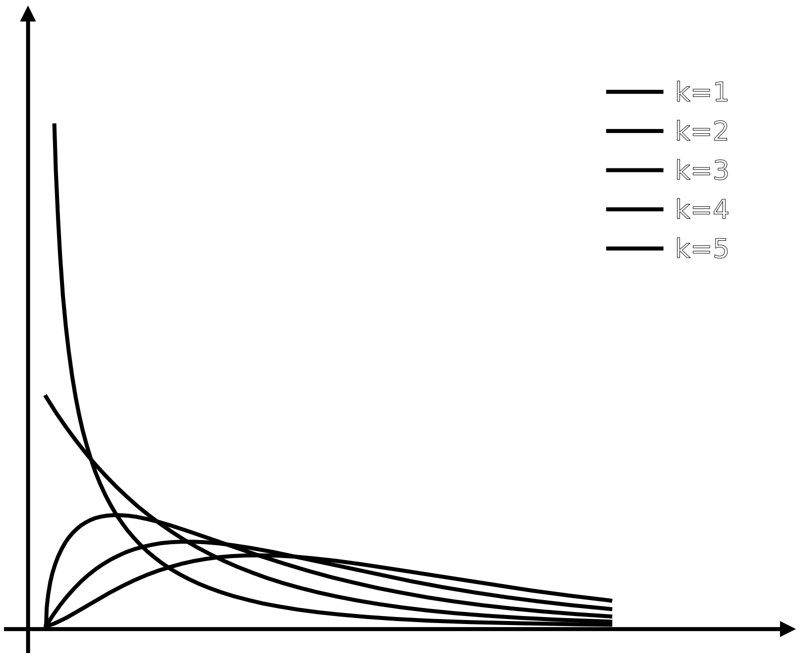
› $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

If the mean or variance changes, the position and shape of the distribution also shift.

We can convert any X that is normally distributed into a **standard normal distribution**, defined as Z, by weighting as follows.

› $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

(2) Chi-square distribution



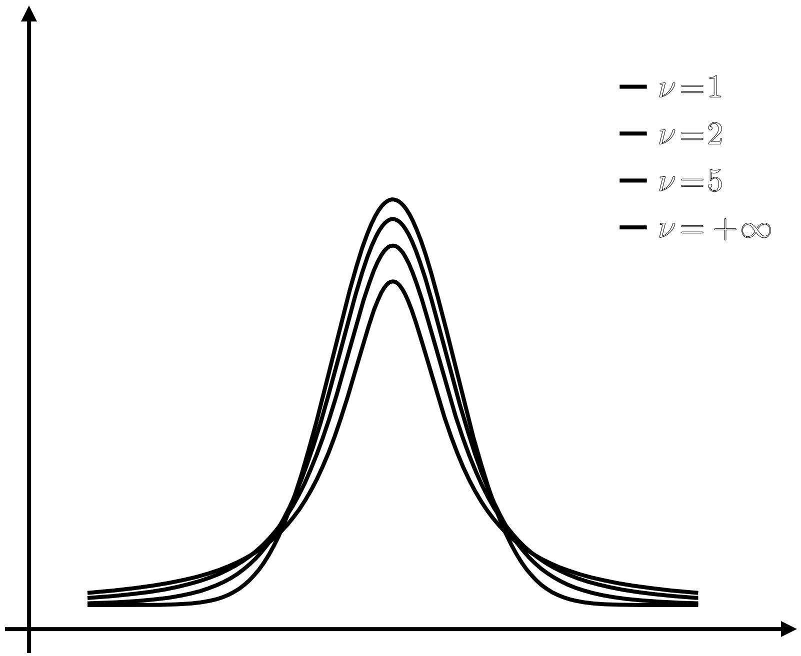
Chi-squared with k degrees of freedom (d.f.) is the distribution of a sum of the squares of k independent standard normal random variables.

› $\chi_k^2 = \sum_{i=1}^{k} Z_i^2$

Propertie of χ_k^2

› Chi-square is skewed depending on d.f. As the d.f. increases it becomes more and more symmetrical.

(3) Student's t-distribution



Let Z and χ_k^2 be random variables distributed as standard normal variable and chi-square respectively and they are independent, the t-distribution with k degrees of freedom can be represented as

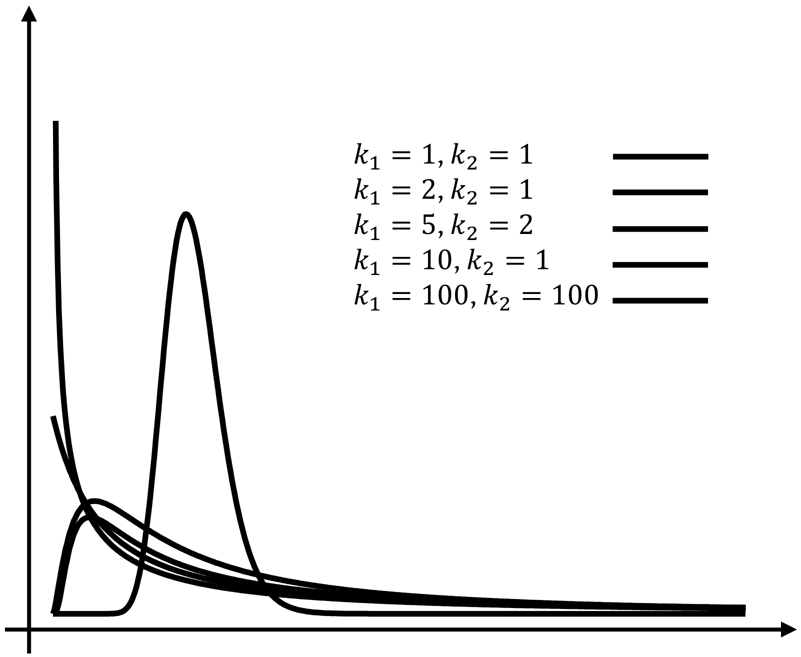
› $t = \frac{Z\sqrt{k}}{\chi_k^2} \sim t_\nu$

<u>Properties of t</u>

› The t-distribution is symmetric but flatter compared to the normal distribution.

› As the d.f. increases, t-distribution is converted to the normal distribution.

(4) F-distribution



Let χ_1^2 and χ_2^2 be random variables distributed as chi-square and they are independent with the d.f. of k_1 and k_2, the F-distribution can be represented as

› $F = \frac{\chi_1^2/k_1}{\chi_2^2/k_2} \sim F(k_1, k_2)$

Properties of F

› The F-distribution is skewed to the right but if k_1 and k_2 becomes larger, the F-distribution becomes normal distribution.

› The square of t-distributed random variable with k degrees of freedom is $t_\nu^2 = F_{1,\nu}$