Solution: Quiz 5

1. Define $H : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}$ as follows:

$$H(x, y) = \left(5y^2,\ 3x - 1 \right) \qquad \text{for } (x, y) \in \mathbb{Z} \times \mathbb{Z}^+,$$

where $\mathbb{Z}^+$ is the set of all positive integers and $\mathbb{Z}$ is the set of all integers.

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Is H one-to-one? Prove or give a counterexample.
Answer: Yes, H is one-to-one. Let $(x_1, y_1), (x_2, y_2)$ be some elements in the domain $\mathbb{Z}\times\mathbb{Z}^+$. We want to show that if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$.
Suppose $H(x_1, y_1) = H(x_2, y_2)$. Then

$$\left(5y_1^2,\ 3x_1 - 1 \right) = \left(5y_2^2,\ 3x_2 - 1 \right)$$

or equivalently, $5y_1^2 = 5y_2^2$ and $3x_1 - 1 = 3x_2 - 1$. I.e.,

$$3x_1 - 1 = 3x_1 - 1 \qquad \Rightarrow \qquad x_1 = x_2$$

and

$$5y_1^2 = 5y_2^2 \quad \Leftrightarrow \quad y_1^2 = y_2^2 \quad \Leftrightarrow \quad y_1 = \pm y_2 \quad \Leftrightarrow \quad y_1 = y_2,$$

(where we have used the fact that both y_1 and y_2 are in $\mathbb{Z}^+$ and so $y_1^2 = y_2^2 \Leftrightarrow y_1 = y_2$ for $y_1, y_2 > 0$). That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$ and therefore, H is one-to-one.

(b) Is H onto? Prove or give a counterexample.
Answer: No, H is not onto. Notice that if we pick (u, v) from the co-domain $\mathbb{Z}^+ \times \mathbb{Z}$ and suppose that there is (x, y) in the domain such that $H(x, y) = (u, v)$, then

$$(u, v) = H(x, y) = \left(5y^2,\ 3x - 1 \right)$$

or we must have

$$u = 5y^2 \quad \Rightarrow \quad y = \pm\sqrt{\frac{u}{5}}$$

and

$$v = 3x - 1 \quad \Rightarrow \quad x = \frac{v + 1}{3}.$$

So, the following is a counterexample for this. If we choose $(u, v) = (1, 1)$ from the co-domain $\mathbb{Z}^+ \times \mathbb{Z}$, then, in order to have $H(x, y) = (1, 1)$, we must set $x = \frac{2}{3}$, and set $y = \pm\sqrt{\frac{1}{5}}$. However, $(x, y) = \left(\frac{2}{3}, -\sqrt{\frac{1}{5}}\right)$, and $\left(\frac{2}{3}, \sqrt{\frac{1}{5}}\right)$ are **not** in the domain $\mathbb{Z} \times \mathbb{Z}^+$ and therefore, we **cannot** find any element (x, y) from the domain $\mathbb{Z} \times \mathbb{Z}^+$ such that $H(x, y) = (1, 1)$. That is, H is not onto.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.
Answer: No, H is not bijective because it is not onto. So we cannot find its inverse function.