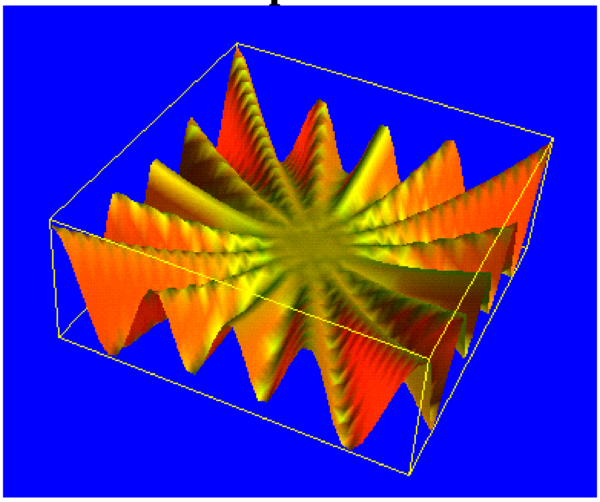
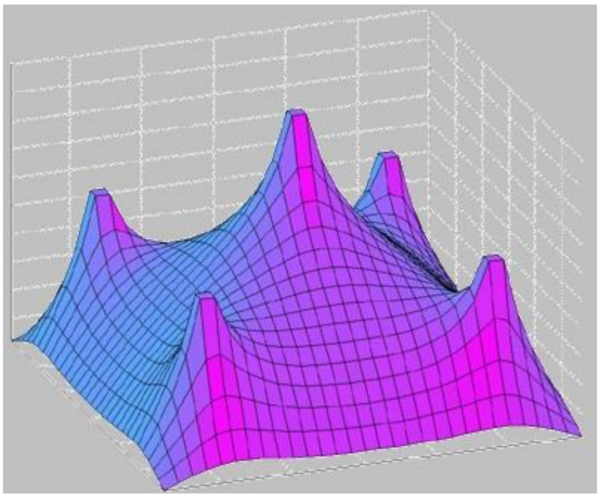
# Optimisation of Multivariable Functions

## 1. Function of Two or More Variables

Typical relationships in economics are relationships of more than one variable, for example,

$$z = f(x, y)$$

$z$ is a function of two variables $x$ and $y$. $z$ is dependent variable (dependent on $x$ and $y$) and $x$ and $y$ are independent variables. It is not stated then the domain is a set of all $(x, y)$ for which the expression $f(x, y)$ is defined (real number).

**Ex.1:**
(a) $f(x, y) = 2x + x^2 y^3$ Find the domain of $f$, $f(1,0)$, $f(0,1)$ and $f(a+1, b)$.
[Ans: $x = \Re, y = \Re$, 2, 0, $2(a+1)+(a+1)^2 b^3$]
(b) $f(x, y) = \dfrac{3x^2 + 5y}{x - y}$ Find the domain of $f$ and $f(1,-2)$.
[Ans: $x \neq y$, -7/3]
(c) $f(x, y) = xe^y + \ln x$ Find the domain of $f$ and $f(e^2, \ln 2)$.
[Ans: $x > 0$, $2(e^2+1) \approx 16.78$
(d) $f(x, y, z) = xy + xz + yz$ Find $f(-1,2,5)$.
[Ans: 3]

**Ex.2:** Production function $F(x, y) = Ax^a y^b$ (Function in this form is called ***'Cobb-Douglas'*** function) where $A, a$ and $b$ are constants. $F$ is number of units produced. $x$ and $y$ are input factors.
Show that

$$F(2x, 2y) = 2^{a+b} F(x, y)$$

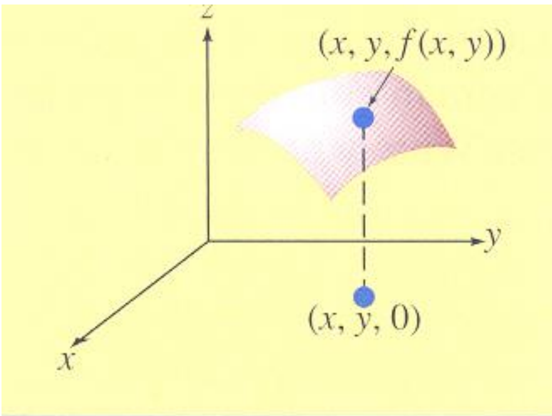
And

$$F(x+h, y) - F(x, y) = Ay^b \{(x+h)^a - x^a\}.$$

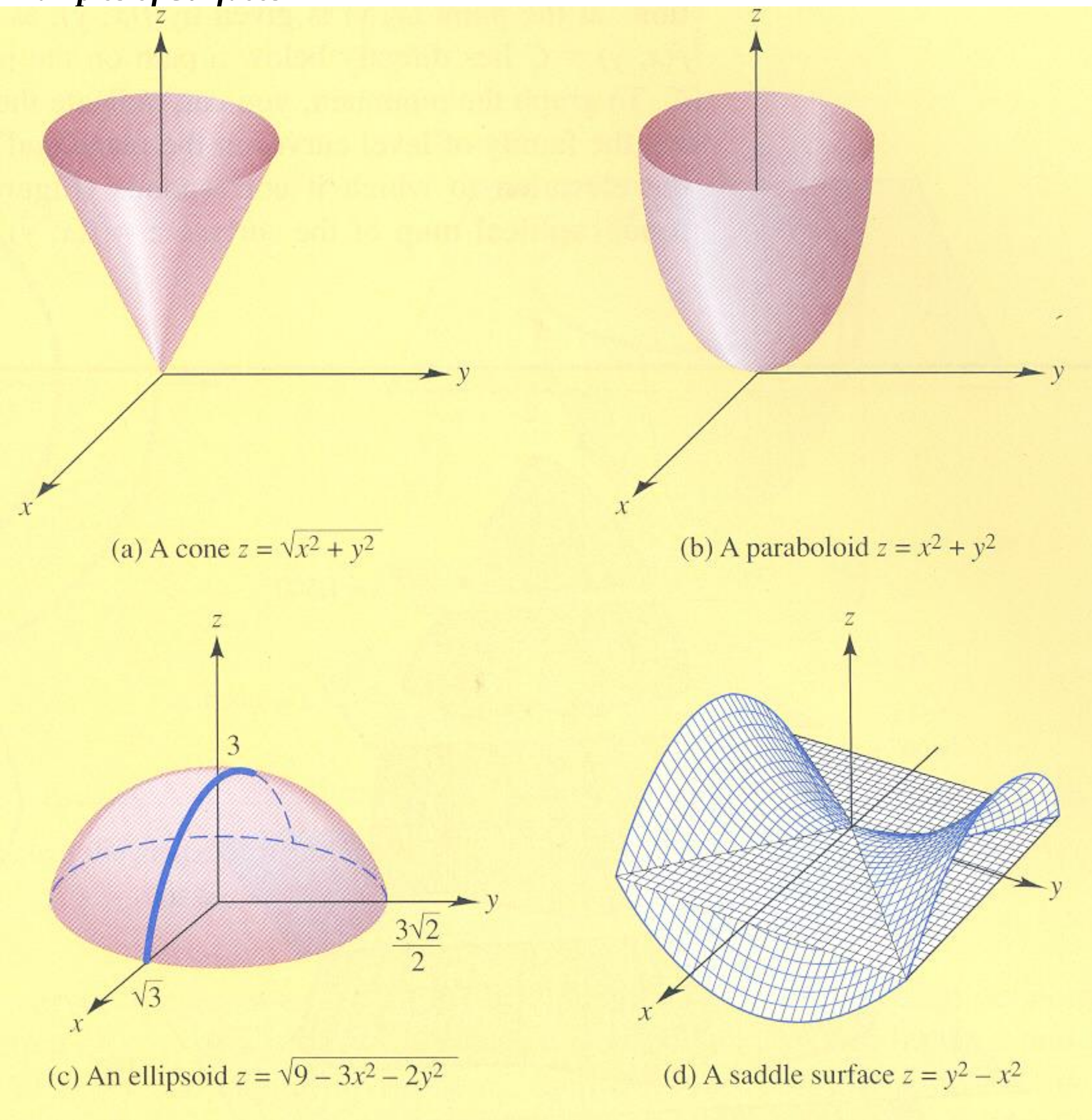
## 2. Geometrical Representation of Function of Two Variables

The graph of a two-variable function is the set of triple $(x, y, z)$ where $(x, y)$ is in the domain of $f$ and $z = f(x, y)$. $z = f(x, y)$ can be plotted in the three-dimensional rectangular coordinate system.

- The origin of the system is where the $x$-, $y$- and $z$-axes intercepts and all three axes are perpendicular.
- The arrows indicate the positive direction.
- Function of two variables can be represented by a 3-D plot.
- Function of more than two variables cannot be visualised.
- The plot of a two-variable function become a surface in three-dimensional space.




***Examples of Surfaces***




(a) A cone $z = \sqrt{x^2 + y^2}$

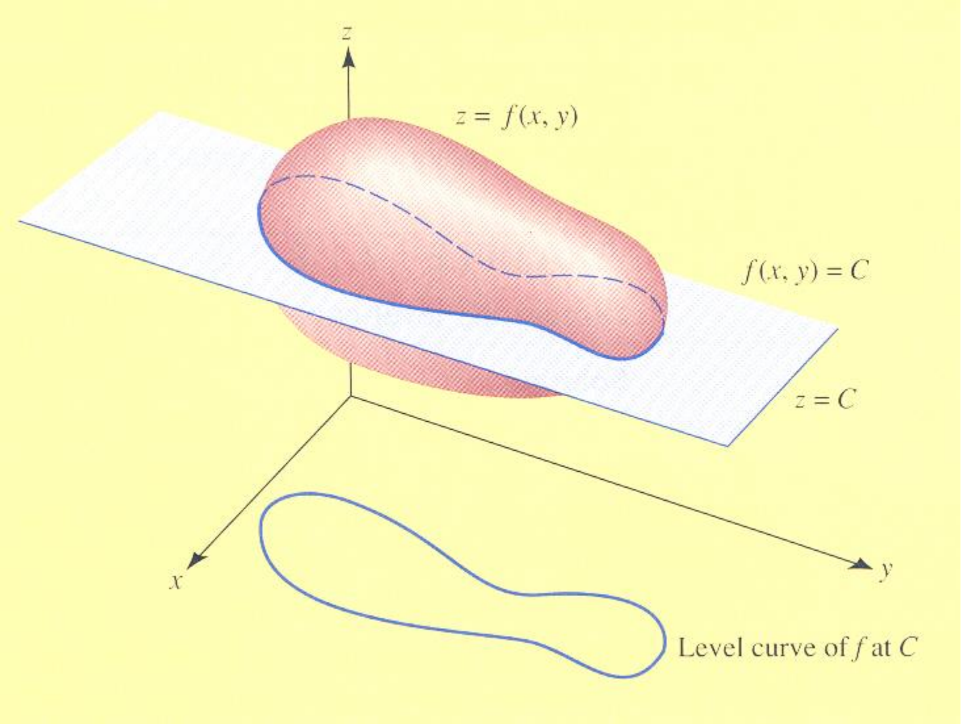
(b) A paraboloid $z = x^2 + y^2$

(c) An ellipsoid $z = \sqrt{9 - 3x^2 - 2y^2}$

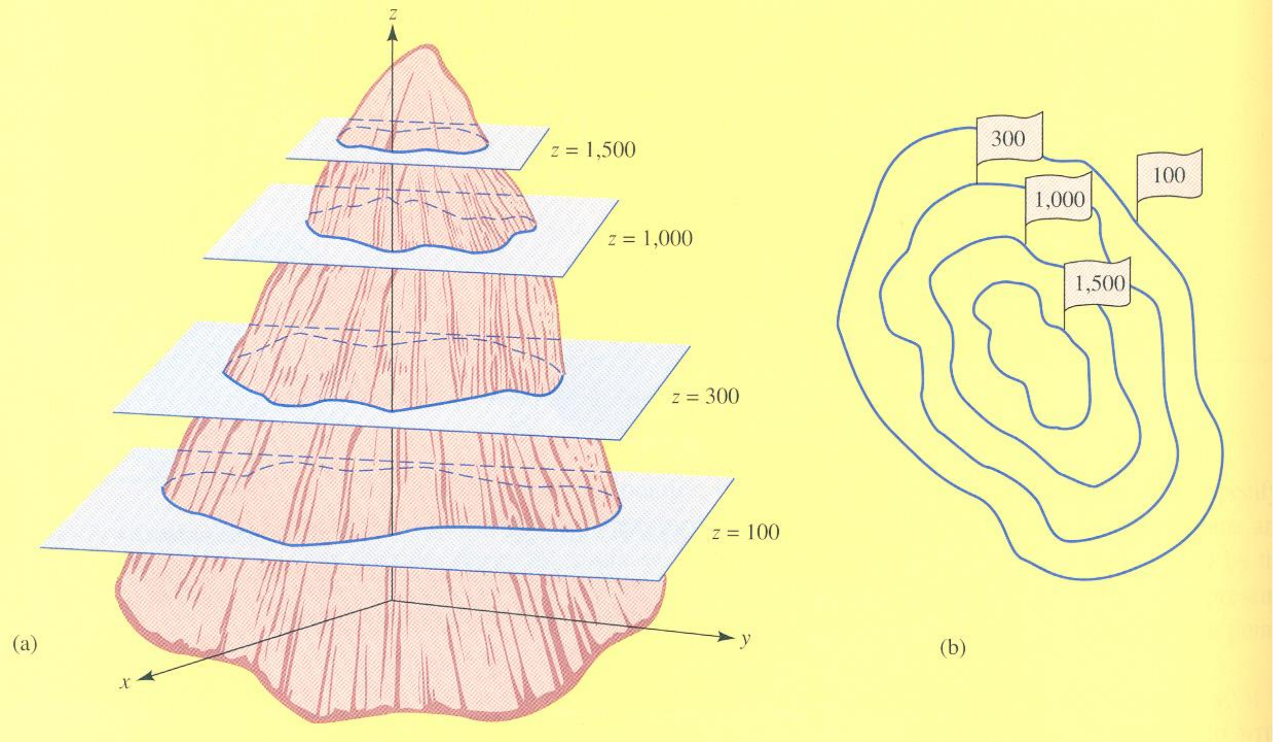
(d) A saddle surface $z = y^2 - x^2$

***Level Curves***:
The set of points $(x, y)$ in the plane *xy* plane that satisfy $f(x, y) = C$ is called the **level curve** of *f* at *C*. An entire family of level curves is generated as *C* varies over a set of numbers. These level curves can help to visualise the surface plot.

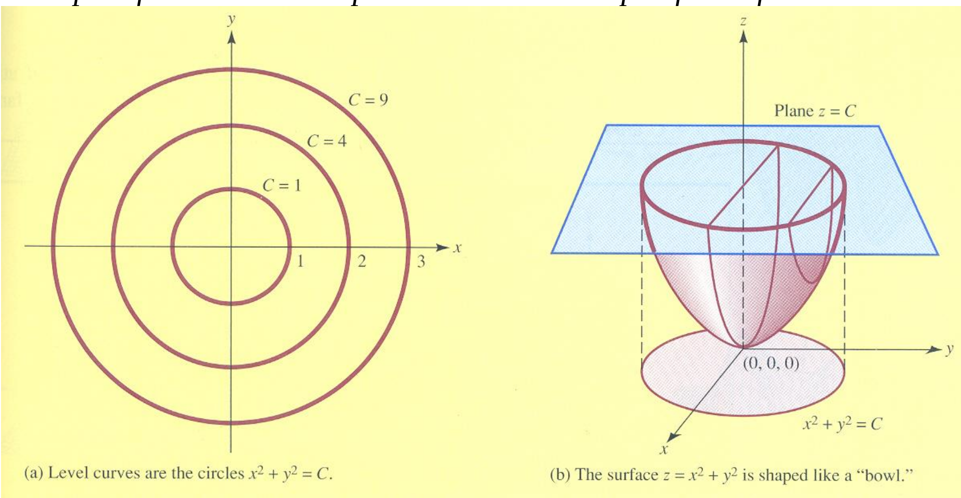
*Example of a level curve*

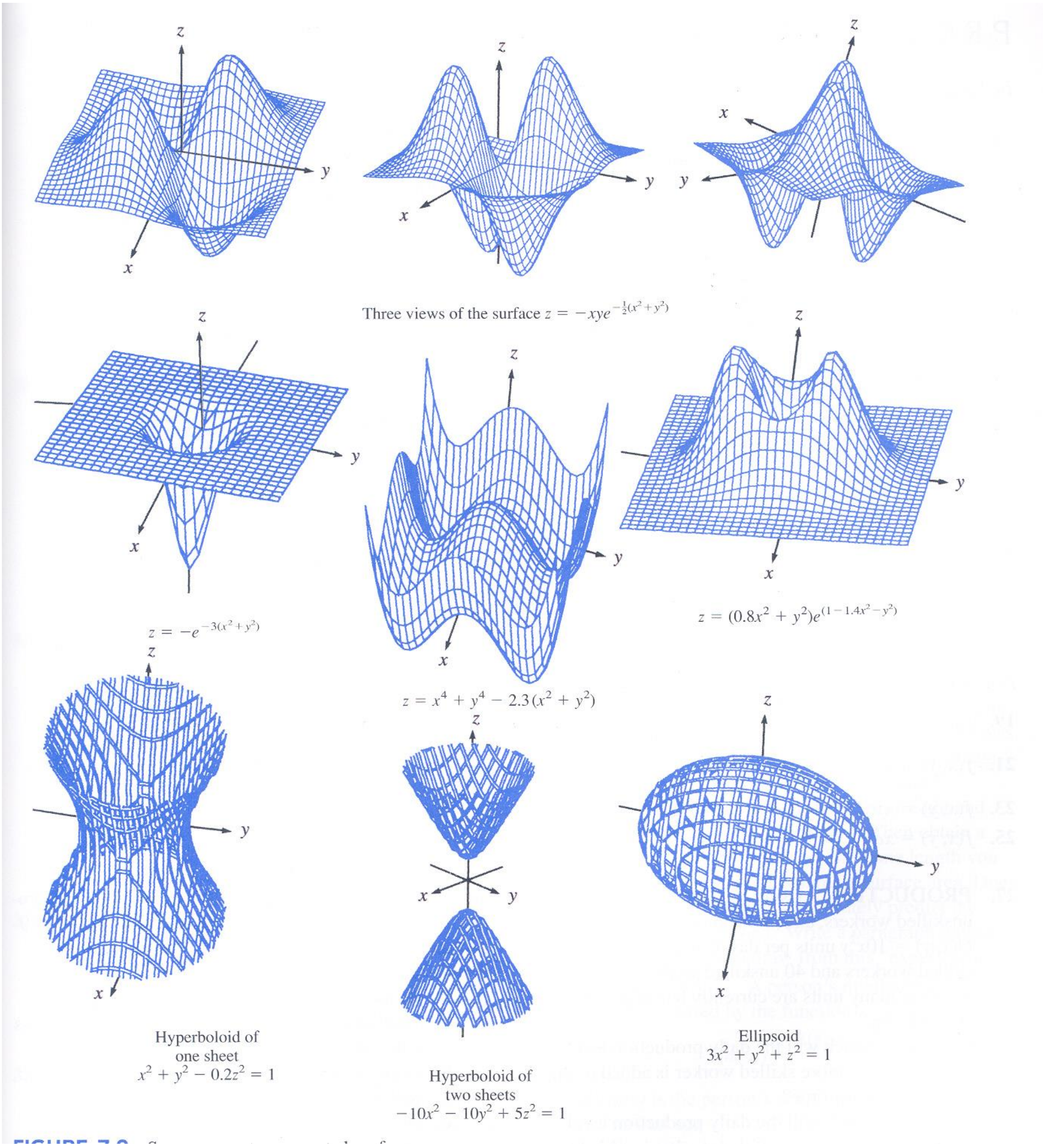



*Example of the application of level curves in a topographical map for mountain*



*Example of level curves help to visualise the shape of a surface*



## 3. Partial Derivatives

Recall a single variable function $y = f(x)$, $\dfrac{dy}{dx}$ is rate of change of $y$ when $x$ changes.
Now, $z = f(x, y)$, if we want to know how $z$ changes when $x$ **or** $y$ changes, we use partial derivatives.

$$z = x^3 + 2y^2$$

When $y$ is constant, **the rate of change of** $z$ **with respect to** $x$**.**

$$\frac{\partial z}{\partial x} = 3x^2$$

**Note:** We use the notation "$\partial$" not "$d$" to indicate partial derivative.
When $x$ is constant, **the rate of change of** $z$ **with respect to** $y$**.**

$$\frac{\partial z}{\partial y} = 4y$$

***Partial Derivative Notations***

- Partial derivative of $f$ or $z$ with respect to $x$ $\quad f_x'(x, y),\ f_x(x, y),\ \dfrac{\partial}{\partial x}\left[f(x, y)\right]$ and $\dfrac{\partial z}{\partial x}$
- Partial derivative of $f$ or $z$ with respect to $x$ evaluated at $(x_0, y_0)$

$$f_x'(x_0, y_0),\ f_x(x_0, y_0), \left[\frac{\partial z}{\partial x}\right]_{(x_0, y_0)}, \left[\frac{\partial z}{\partial x}\right]_{\substack{x=x_0\\ y=y_0}}, \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} \text{ and } \left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}}$$

**Ex.3**: $f(x, y) = x^3y + x^2y^2 + x + y^2$ Find all partial derivatives of $f(x, y)$**.**

$$\frac{\partial f}{\partial x} =$$
$$\frac{\partial f}{\partial y} =$$

**Ex.4:**
(a) $f(x, y) = \dfrac{xy}{x^2 + y^2}$ Find all partial derivatives of $f(x, y)$**.**

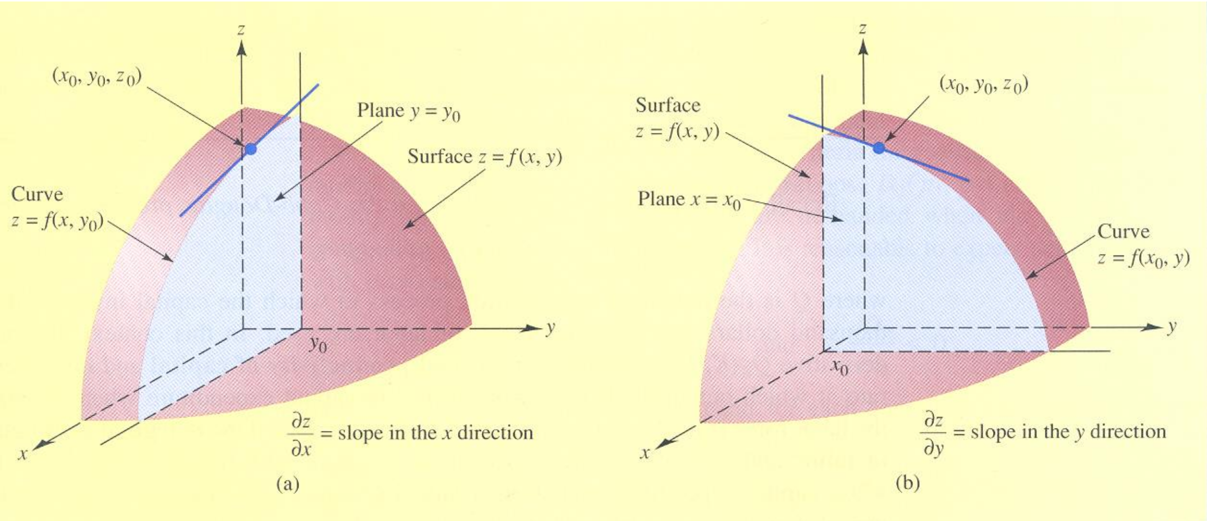
[Ans: $\dfrac{\partial f}{\partial x} = \dfrac{y^3 - x^2y}{\left(x^2 + y^2\right)^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{x^3 - xy^2}{\left(x^2 + y^2\right)^2}$ ]

(b) $z = \left(x^2 + xy + y\right)^5$ Find all partial derivatives.

[Ans: $\dfrac{\partial z}{\partial x} = 5\left(x^2 + xy + y\right)^4(2x + y)$ and $\dfrac{\partial z}{\partial y} = 5\left(x^2 + xy + y\right)^4(x + 1)$]

**Ex.5:** A demand for rice is $x = \dfrac{Am^{2.08}}{p^{1.5}}$ where $x$ is the rice consumption, $m$ is the income per family, $p$ is the price, and $A$ is a constant. Calculate $\dfrac{\partial x}{\partial p}$ and $\dfrac{\partial x}{\partial m}$.

**4. Formal Definition of Partial Derivatives**




Slope of tangent line at $(x_0, y_0, f(x_0, y_0))$ = partial derivative of $z$ (or $f$) with respect to $x$ at $(x_0, y_0)$

By definition, $$\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} = f_x'(x_0,y_0) = \lim_{h\to 0}\left(\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\right)$$

Similarly, $$\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} = f_y'(x_0,y_0) = \lim_{h\to 0}\left(\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}\right)$$

**Ex.6:** A company produces 2 products, the joint cost function per week is given by $C = f(x,y) = 0.07x^2 + 75x + 85y + 6000$. Determine marginal cost $\dfrac{\partial C}{\partial x}$ and $\dfrac{\partial C}{\partial y}$ when $x = 100$ and $y = 50$. [Ans: 89,85]

**Solution:** $\dfrac{\partial C}{\partial x} =$ $\qquad$ $\dfrac{\partial C}{\partial y} =$

$\left[\dfrac{\partial C}{\partial x}\right]_{(100,50)} =$ $\qquad$ $\left[\dfrac{\partial C}{\partial y}\right]_{(100,50)} =$

## 5. Implicit Partial Differentiation

$z$ is not given in term of $x$ and $y$, for example,

$$z^2 - x^2 - y^2 = 0 \qquad \text{Find } \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

Partial differentiate both sides of the equation with respect to $x$ to find $\frac{\partial z}{\partial x}$.

$$2z\frac{\partial z}{\partial x} - 2x + 0 = 0$$

Rearranging, $$\frac{\partial z}{\partial x} = \frac{x}{z}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y}{z}$$

**Ex.7:** If $z = f(x,y)$ and $e^{yz} = -xyz$, find $\frac{\partial z}{\partial x}$ where $x = -\frac{e^2}{2}$, $y = 1$ and $z = 2$. [Ans: $-\frac{4}{e^2}$]

## 6. Higher-Order Partial Derivative

For $z = f(x,y)$,

1st order partial derivatives are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

2nd order partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

**Note:** Beyond the 2nd order derivatives can also be determined e.g.

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right] = \frac{\partial^3 f}{\partial y \partial x^2} = f_{xxy}$$

**Ex.8:** For $f(x,y) = 7x^2 + 3y$, find $f_y$, $f_{yy}$ and $f_{yyx}$. [Ans: 3, 0, 0]

**Ex.9:** For $f(x,y) = (x+y)^2(xy)$, find $f_x$, $f_y$, $f_{xx}$, $f_{yy}$, $f_{yyx}$ and $f_{xyy}$.
[Ans: $3x^2y + 4xy^2 + y^3, x^3 + 4x^2y + 3xy^2, 6xy + 4y^2, 4x^2 + 6xy, 8x + 6y, 8x + 6y$ ]

**Ex.10:** For $f(x,y) = y^2e^x + \ln(xy)$, find $f_{xyy}(1,1)$. [Ans: 2$e$]

## 7. Chain Rule

$$z = f_1(x,y),\ x = f_2(r,s) \text{ and } y = f_3(r,s)$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} \qquad \text{and} \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

**Note:** In the chain rule, the number of intermediate variables of $z$ (in this case is 2) is the same as the number of terms that compose each of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$.

**Ex.11:** For $z = 5x + 3y$, $x = 2r + 3s$ and $y = r - 2s$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$. [Ans: 13,9]

**Ex.12:** $C$ is the total cost of producing $q_C$ units of digital cameras and $q_F$ units of memory sticks is

$$C = 30q_C + 0.015q_C q_F + q_F + 900.$$

The demand function is $q_C = \frac{9000}{p_C\sqrt{p_F}}$ and $q_F = 2000 - p_C - 400p_F$ where $p_C$ is the price per digital camera and $p_F$ is the price per memory sticks. Find the rate of change of the total cost with respect to price of digital camera when the price per digital camera is 50 and the price per memory stick is 2. [Ans: -123.2]

## 8. Chain Rule (Extension)

Similar to the chain rule for single-variable functions,

- If $z = f(x,y)$, $x = g_1(r,s,t)$ and $y = g_2(r,s,t)$ then

  The partial derivative of $z$ with respect to $r$ is $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$

  The partial derivative of $z$ with respect to $s$ is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$

  The partial derivative of $z$ with respect to $t$ is $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$

- If $z = f(w,x,y)$, $w = g_1(r,s)$, $x = g_2(r,s)$ and $y = g_3(r,s)$ then

  The partial derivative of $z$ with respect to $r$ is $\frac{\partial z}{\partial r} =$

  The partial derivative of $z$ with respect to $s$ is $\frac{\partial z}{\partial s} =$

- **General case**
  If $z = f(x_1, x_2, ..., x_n)$, $x_1 = g_1(y_1, y_2, ..., y_m)$, $x_2 = g_2(y_1, y_2, ..., y_m)$ and $x_n = g_n(y_1, y_2, ..., y_m)$ where *m* and *n* are positive integers then there are *m* possible partial derivatives of *z* which are

$$\frac{\partial z}{\partial y_1} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_1} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial y_1} + ... + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial y_1}$$

$$\frac{\partial z}{\partial y_2} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_2} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial y_2} + ... + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial y_2}$$

$$\vdots$$

$$\frac{\partial z}{\partial y_m} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_m} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial y_m} + ... + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial y_m}$$

The change in $y_i$ affects $x_i$ and subsequently affects $z$.

**Ex.13:** If $z = \frac{x + e^y}{y}$, $x = rs + se^{rt}$ and $y = 9 + rt$, find $\frac{\partial z}{\partial s}$ when $r = -2$, $s = 5$ and $t = 4$.
[Ans: $-2 + e^{-8}$]

## 9. Young's Theorem

All the $m^{\text{th}}$-order partial derivatives of the function $f(x_1, x_2, ..., x_n)$ are continuous. If any two involve differentiating with respect to each of the same variables the same number of times, then they are necessary equal. For example, if $m = 2$ and $z = f(x, y)$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

Provided that slops are continuous.

**Ex.14:** For $z = f(x, y) = 2x^4 + 3x^3y^3 + xy^2 + y$, show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 27x^2y^2 + 2y$.

## 10. Competitive /Complementary Products

For related products, a price change of one affects the demand of the others.

- If an increase in demand for one product results in a decrease in demand for another product, the two products are said to be ***COMPETITIVE*** or ***SUBSTITUTE products***. Examples of competitive or substitute products are Coke and Pepsi; eye glasses and contact lens, ______________________________________________

- If an increase in demand for one product results in an increase in demand for another product, the two products are said to be ***COMPLEMENTARY products***. Examples of complementary products are eye glasses and lens, digital camera and memory sticks, tennis rackets and tennis balls,
  ______________________________________

Partial derivatives can be used to determine whether two products are competitive, complementary, or neither. The demand functions for two products are dependent on the prices for both products $p_A$ and $p_B$ :-

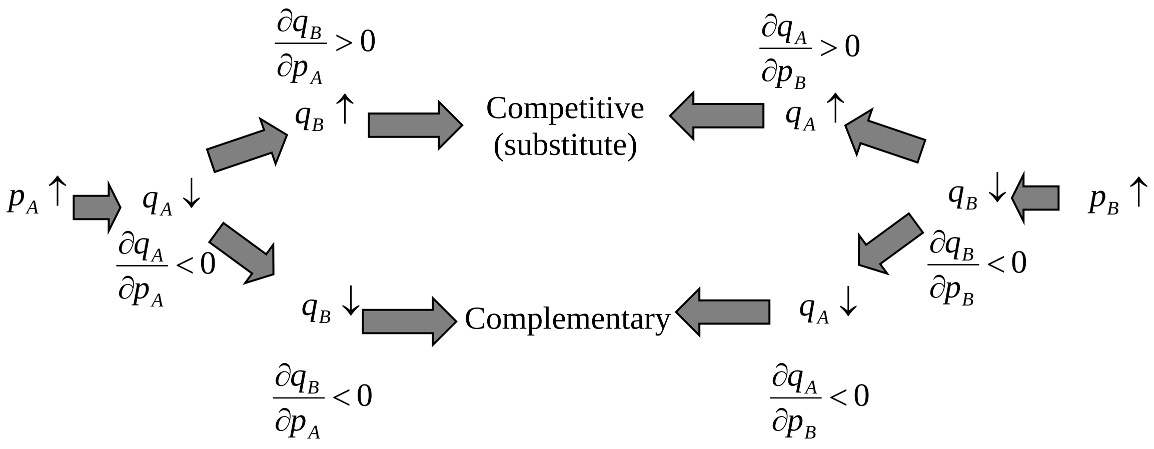
| | |
|---|---|
| $q_A = f(p_A, p_B)$ | Demand function for product *A* |
| $q_B = g(p_A, p_B)$ | Demand function for product *B* |

Normally, if the price of *A* increases while the price of *B* is held constant, the demand for *A* will decrease; that is $\dfrac{\partial q_A}{\partial p_A} < 0$ (i.e. marginal demand of *A* with respect to the price of *A*). In addition, if *A* and *B* are competitive products, the demand for *B* will increase; that is, $\dfrac{\partial q_B}{\partial p_A} > 0$.

Similarly, if the price of *B* increases while the price of *A* is held constant, the demand for *B* will decrease; that is $\dfrac{\partial q_B}{\partial p_B} < 0$ (i.e. marginal demand of *B* with respect to the price of *B*). In addition, if *A* and *B* are competitive products, the demand for *A* will increase; that is, $\dfrac{\partial q_A}{\partial p_B} > 0$.

Complementary products have similar reasoning, hence, $\dfrac{\partial q_B}{\partial p_A} < 0$ and $\dfrac{\partial q_A}{\partial p_B} < 0$.

| **Product *A* and *B*** | **Partial Derivatives** |
|---|---|
| Competitive (substitute) | $\dfrac{\partial q_A}{\partial p_B} > 0$ and $\dfrac{\partial q_B}{\partial p_A} > 0$ |
| Complementary | $\dfrac{\partial q_B}{\partial p_A} < 0$ and $\dfrac{\partial q_A}{\partial p_B} < 0$ |
| Neither | $\dfrac{\partial q_B}{\partial p_A} \ge 0$ and $\dfrac{\partial q_A}{\partial p_B} \le 0$ |
| Neither | $\dfrac{\partial q_B}{\partial p_A} \le 0$ and $\dfrac{\partial q_A}{\partial p_B} \ge 0$ |



**Ex.15:** Determine whether the indicated products are competitive, complementary or neither where their demand functions are.

a) $q_A = 8000 - 0.09{p_A}^2 + 0.08{p_B}^2$ and $q_B = 15000 + 0.04{p_A}^2 - 0.3{p_B}^2$

b) $q_A = 200 - 5p_A + 4{p_B}^2$ and $q_B = 300 + 2p_A - 4p_B$

c) $q_A = 800 - 0.004{p_A}^2 - 0.003{p_B}^2$ and $q_B = 600 - 0.003{p_A}^2 - 0.003{p_B}^2$

d) $q_A = 500 - 0.5p_A - {p_B}^2$ and $q_B = 10000 - 0.5p_A - 100{p_B}^2$

[Ans: (a) Competitive, (b) competitive, (c) complementary, and (d) complementary]

**Ex.16:** The weekly demand of *A* and *B* are $q_A = e^{-\left(\frac{p_A}{p_B}\right)}$ and $q_B = \dfrac{16}{p_A {p_B}^2}$ where $q_A$ and $q_B$ are the demand of *A* and *B* respectively, $p_A$ and $p_B$ are price per unit in thousand of baht of *A* and *B* respectively.

a) *A* and *B* are competitive, complementary or neither.
b) If unit prices of *A* and *B* are 1,000 baht and 2,000 baht respectively, estimate demand change in *A* when the price of *A* is held constant and the price of *B* decreases by 40 baht. [Ans: $-0.01/\sqrt{e}$ ]

**11. Elasticity (Extension)**

Recall, the single-variable demand function,

$$E = \frac{\%\,\text{change in demand}}{\%\,\text{change in price}} = \frac{p}{q} \cdot \frac{dq}{dp}$$

| $E(q)$ | **Demand** | **Interpretation** |
|---|---|---|
| $0 < \lvert E(q) \rvert < 1$ | Inelastic | Demand is not sensitive to changes in price. A given change in price produces a smaller change in demand. |
| $\lvert E(q) \rvert > 1$ | Elastic | Demand is sensitive to changes in price. A change given in price produces a larger change in demand. |
| $\lvert E(q) \rvert = 1$ | Unit | A given change in price produces the same change in demand. |

**Ex.17:** In a supermarket, rice has the elasticity of -1.1 and potato has the elasticity of -0.2. Which product is more essential to customers?

**Ex.18:** A demand function for a product is $q = 8000p^{-1.5}$. Find the elasticity and percentage change in demand when the price of the product is increased by 1% from the price of 4.

[Ans: $E = -1.5$, 1.5% decreases in demand]

**Partial Elasticity**

If the demand of a product is a function of two variables (price $x$ and price $y$) i.e. $q = f(x, y)$, the elasticity of this product can be defined using partial differentiation with respect to each variable.

The percentage change in demand with respect to the percentage change in variable $x$:

$$E_{qx} = \frac{x}{q}\left(\frac{\partial q}{\partial x}\right)$$

The percentage change in demand with respect to the percentage change in variable $y$:

$$E_{qy} = \frac{y}{q}\left(\frac{\partial q}{\partial y}\right)$$

**Ex.19:** A product has the demand function $z = Ax^a y^b$. Show that the partial elasticity of $z$ with respect to $x$ is $E_{zx} = a$.

$$E_{zx} = \frac{x}{z}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{z}\left(Ay^b \cdot ax^{a-1}\right) = \frac{aAx^a y^b}{z}$$

$$E_{zx} = \frac{a}{z}\left(Ax^a y^b\right) = \frac{a}{z}(z) = a$$

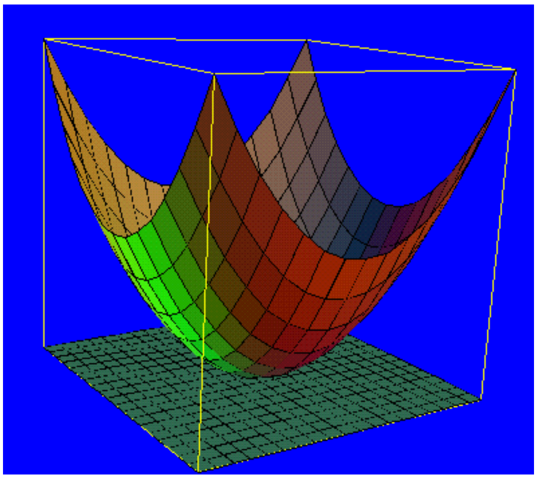
Similarly, the partial elasticity with respect to $y$ is $E_{zy} = b$.

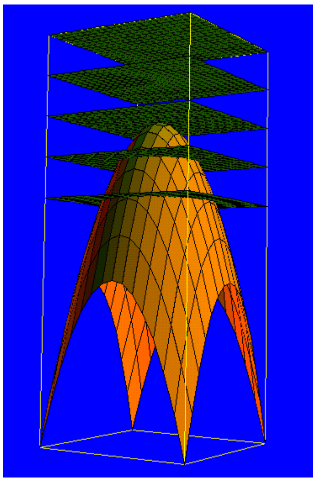
**Ex.20:** The annual demand function of rice in Thailand is estimated to be $q_r = Ap_r^{-0.28} m^{0.34}$ where $p_r$ is the price of rice per kilogram and $m$ is the average income of the people per year. The annual demand for orange is $q_o = Bp_o^{-1.27} m^{1.32}$ where $p_o$ is the price of orange per kg. Show that the elasticity of demand with respect to the price and income are

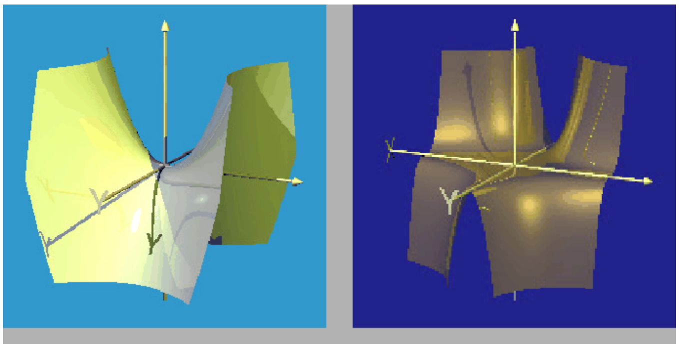
$$E_{q_r p_r} = -0.28, E_{q_o p_o} = -1.27, E_{q_r m} = 0.34, E_{q_o m} = 1.32.$$

Explain the significance of each elasticity.

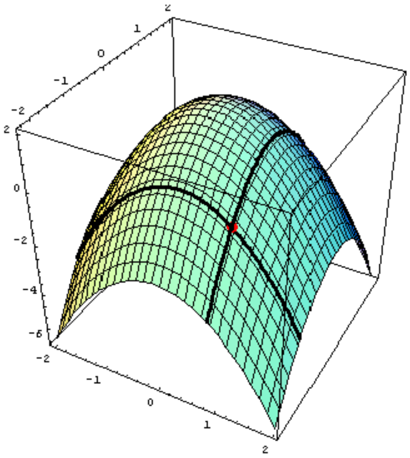
## 12. Multivariable Optimisation

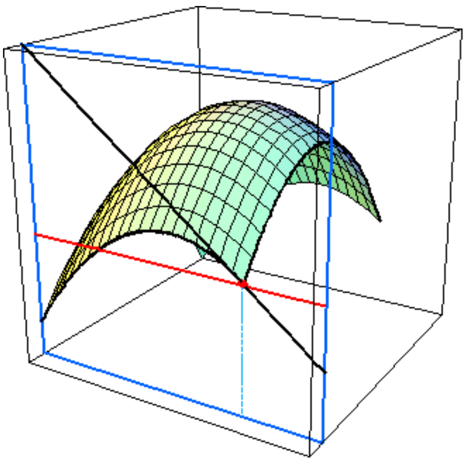


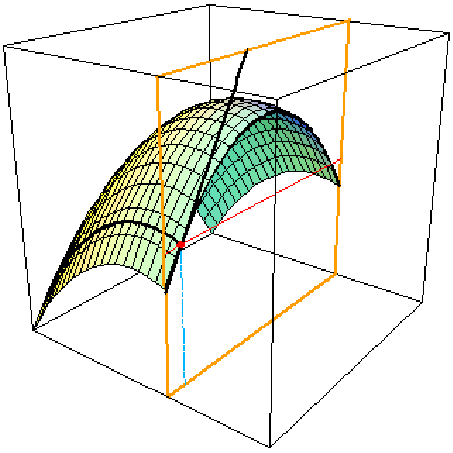




(q) $z = x^2 - y^2$ (r) $z = x^2 y^3$







- $f(x_0, y_0)$ is a relative maximum if $f(x_0, y_0) \geq f(x, y)$ for all $x$ and $y$ closed to $(x_0, y_0)$.
- $f(x_0, y_0)$ is a relative minimum if $f(x_0, y_0) \leq f(x, y)$ for all $x$ and $y$ closed to $(x_0, y_0)$.

**Necessary Conditions for Relative Extrema**

A two-variable function, $z = f(x,y)$, has a relative maximum or minimum at $(x_0, y_0)$ if the point $(x_0, y_0)$ satisfies

$$f_x'(x,y) = 0 \quad \left(\frac{\partial z}{\partial x} = 0\right) \qquad \text{and} \qquad f_y'(x,y) = 0 \quad \left(\frac{\partial z}{\partial y} = 0\right)$$

Hence, $(x_0, y_0)$ is a critical point if $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$.

**Ex.21:** $z = f(x,y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$ Assume that $z$ has a relative maximum point, find this point. [Ans:(5,8)]

**Second-Derivative Test for Relative Extrema**

Let $z = f(x,y)$ has continuous $f_{xx}''(x,y), f_{yy}''(x,y)$ and $f_{xy}''(x,y)$ at all points $(x,y)$ near the critical point $(x_0, y_0)$. Let $D$ be the function defined by

$$D(x,y) = f_{xx}''(x,y)\, f_{yy}''(x,y) - \left[f_{xy}''(x,y)\right]^2$$

- If $f_{xx}''(x,y) < 0$ and $D(x_0, y_0) > 0$, $f$ has a relative maximum at $(x_0, y_0)$.
- If $f_{xx}''(x,y) > 0$ and $D(x_0, y_0) > 0$, $f$ has a relative minimum at $(x_0, y_0)$.
- If $D(x_0, y_0) < 0$, $f$ has neither a relative maximum nor minimum at $(x_0, y_0)$. This point $(x_0, y_0)$ is known as a **saddle point**.
- If $D(x_0, y_0) = 0$, no conclusion about an extremum at $(x_0, y_0)$ can be drawn, and further analysis is required..

**Ex.22:** $f(x,y) = x^3 - x^2 - y^2 + 8$ Find all critical points and classify them.

$$f_x'(x,y) = 3x^2 - 2x = 0 \qquad \Rightarrow \qquad x = 0 \quad \text{or} \quad x = \frac{2}{3}$$

$$f_y'(x,y) = -2y = 0 \qquad \Rightarrow \qquad y = 0$$

Hence, critical points are (0, 0) and $\left(\dfrac{2}{3}, 0\right)$.

$$f_{xx}''(x,y) = 6x - 2$$

$$f_{yy}''(x,y) = -2$$

$$f_{xy}''(x,y) = 0$$

Consider the critical point (0, 0),

$$f_{xx}''(0,0) = -2 < 0, \quad f_{yy}''(0,0) = -2, \quad f_{xy}''(0,0) = 0, \quad D(0,0) = (-2)(-2) - 0^2 = 4 > 0$$

$\therefore$ $(0,0)$ is a relative maximum.

Consider the critical point $\left(\frac{2}{3},0\right)$,

$$f_{xx}''\left(\frac{2}{3},0\right)=6\left(\frac{2}{3}\right)-2=2>0, \quad f_{yy}''\left(\frac{2}{3},0\right)=-2, \quad f_{xy}''\left(\frac{2}{3},0\right)=0,$$

$$D\left(\frac{2}{3},0\right)=(2)(-2)-0^2=-4<0$$

$\therefore \left(\frac{2}{3},0\right)$ is a saddle point.

**Ex.23:** $f(x,y)=\frac{x^3}{3}+y^2-2x+2y-2xy$ Determine all critical points and classify them.
[Ans: $f_x'(x,y)=x^2-2-2y=0$, $f_y'(x,y)=2y+2-2x=0$, $(0,-1)$ saddle point, $(2,1)$ rel. min.]

**Ex.24:** A production function of a firm is $p=1.08l^2-0.03l^3+1.68k^2-0.08k^3$. Find quantities of *l* and *k* which maximise the output *p*.
[Ans: Critical points are (0,0), (0,14), (24,0), (24,14). Maxima *l* = 24 and *k* = 14]

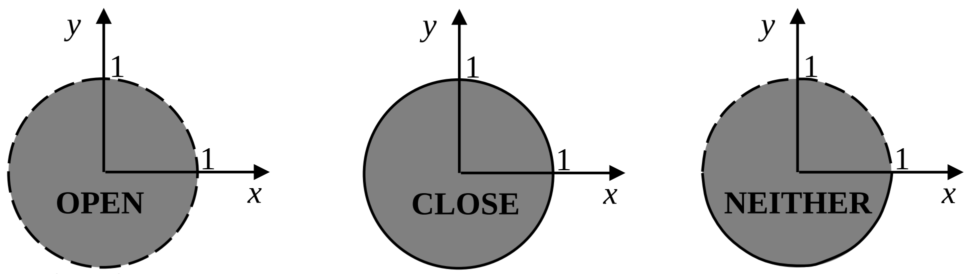
## 13. Optimisation of Multivariable Function with Constraints

Three methods will be introduced:-

1. Extreme Value Theorem for closed and bounded inequality constraints
2. Substitution Method for equality constraints
3. Lagrange Multiplier Method for any constraints

### 14.1 Domain – Range of consideration

1-variable function ➔ the domain is an interval
2-variable function ➔ the domain is a plane (set)




$x^2+y^2<1$ $\qquad$ $x^2+y^2\le 1$

**Ex.25:** Classify if the following sets are open, close or neither.
a) $2x+y\le 2, x\ge 0$ and $y\ge 0$
b) $2x+y<2, x>0$ and $y>0$
c) $2x+y\le 2, x>0$ and $y\ge 0$

| A set (domain) is considered bounded if the whole set is contained within a closed area. |
| --- |

**Ex.26:** Determine if the following sets is bounded or not.
a) $x \geq 1$ and $y \geq 0$
b) $4 \leq x^2 + y^2 \leq 9$
c) $4 < x^2 + y^2 \leq 9$
d) $x^2 + y^2 \geq 9$

### 14.2 Extreme-Value Theorem for Two-Variable Function

The extrema-value theorem for one-variable function can be generalised into multi-variable function. If a function $f(x, y)$ is continuous throughout **a closed bounded set *S*** in a plane, then there exists both a point $(a,b)$ in *S* where $f(a,b)$ has an absolute minimum and a point $(c,d)$ is *S* where $f(c,d)$ has an absolute maximum, i.e.

$$f(a,b) \leq f(x,y) \leq f(c,d) \quad \text{for all } (x,y) \text{ in } S.$$

This is sufficient but not necessary.

**To find maxima and minima** of a differentiable function $f(x, y)$ defined on a closed bounded set *S* in a plane:

(1) Find all critical points of $(x, y)_{c,i}$ in *S* by differentiating with respect to each variable and equating to zero. $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

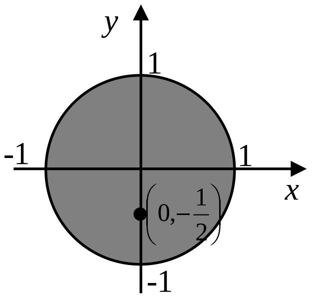
(2) If asked, classify relative maximum and minimum points. **(ONLY if ASKED)**

(3) Compare relative maximum and minimum points to find absolute maximum and minimum points by comparing

a. the **critical points** of $f(x, y)$ on **the boundary of *S*** and the **corner points** (same as in one-variable case).
b. the values of $f(x, y)_{c,i}$ in (1)

**Ex.27:** Find absolute maxima and minima of $f(x, y)$ defined over *S* when
$z = f(x, y) = x^2 + y^2 + y - 1$ subject to $S = \{(x, y) : x^2 + y^2 \leq 1\}$

**Procedures:**




(1) Find critical points

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x = 0 \quad \Rightarrow \quad x = 0 \\ \frac{\partial f}{\partial y} &= 2y + 1 = 0 \quad \Rightarrow \quad y = -\frac{1}{2} \end{aligned}\right\}$$

1st order condition gives **ONLY** one critical point $\left(0, -\frac{1}{2}\right)$.

(2) Don't need to classify the nature of the critical point.

(3) (a) Find the absolute maximum or minimum points on the boundary $S: x^2 + y^2 = 1$
The domain of $x$ is [-1, 1] and the domain of $y$ is also [-1, 1] and $x^2 + y^2 = 1$.
Hence, on the boundary of $S$, $f(x,y)$ becomes

$$z = f(x,y) = \underbrace{x^2 + y^2}_{} + y - 1 = 1 + y - 1 \quad \Rightarrow f(x,y) = y \qquad \text{and} \quad y \in [-1,1]$$

Hence, $f(x,y)$ is **maximised** when $y = 1$ and $x = 0$, $f(0,1) = 1$
$f(x,y)$ is **minimised** when $y = -1$ and $x = 0$, $f(0,-1) = -1$

(b) Calculate the value of the critical point $\left(0, -\frac{1}{2}\right)$.

$$f\left(0, -\frac{1}{2}\right) = 0^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 1 = -\frac{5}{4}$$

**Compare (a) and (b):** $f(0,1) = 1$, $f(0,-1) = -1$ and $f\left(0, -\frac{1}{2}\right) = -\frac{5}{4}$.

**Hence,** maximum value of $f(x,y)$ in $S$ is 1 at (0, 1).
Minimum value of $f(x,y)$ in $S$ is $-\frac{5}{4}$ at $\left(0, -\frac{1}{2}\right)$.

**Ex.28.a:** For a two-variable function,

$$f(x,y) = x^2 - xy + y^2 + 4$$

(a) Find critical point(s) using the first derivative. [Ans: (0,0)]

(b) Find the maximum and minimum value of the above function $f(x,y)$ on the closed area in the first quadrant bounded by the triangle formed by the line $x = 0$, $y = 4$ and $y = x$.
[Ans: Maximum value $f(0,4)$ = $f(4,4)$ = 20 and minimum value $f(0,0) = 4$]

**Ex.28.b:** Thai government is promoting transportation vehicles e.g. taxi, van and trucks to use natural gas (NG). Currently the government is importing NG from Burma ($x$) and Malaysia ($y$). The benefit function of import NG is given by $f(x,y) = 9x + 8y - 6(x+y)^2$. Due to the limitation of capacity $0 \le x \le 5$ and $0 \le y \le 3$. For potential reason, importing from Burma should not be too small, so that $x \ge 2(y-1) \quad \Rightarrow \quad y \le \frac{x}{2} + 1$. Find the import value from Burma and Malaysia that will maximise the benefit.

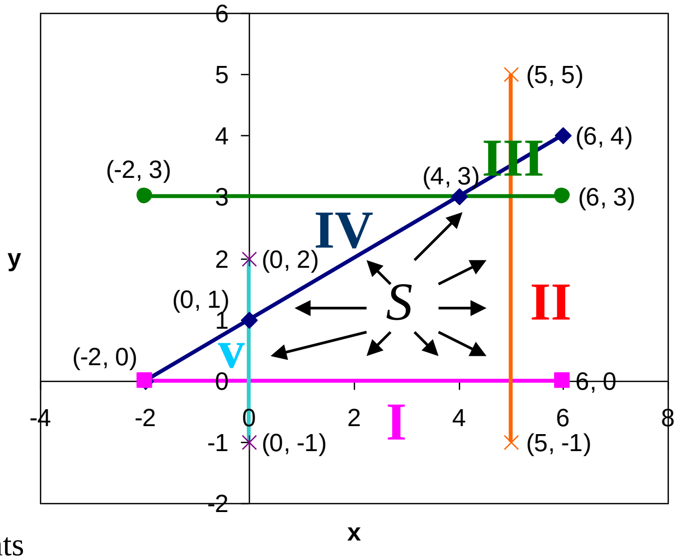
| |
|---|
| Max. $f(x,y) = 9x + 8y - 6(x+y)^2$ subject to $0 \le x \le 5$; $0 \le y \le 3$ and $y \le \frac{x}{2} + 1$ |

$$\text{Max. } f(x,y) = 9x + 8y - 6(x+y)^2 \text{ subject to } 0 \le x \le 5;\ 0 \le y \le 3 \text{ and } y \le \frac{x}{2} + 1$$

I: $y \ge 0$
II: $x \le 5$
III: $y \le 3$
IV: $2y - x \le 2$
V: $x \ge 0$

(I)+(III) ➔ $0 \le y \le 3$
(II)+(V) ➔ $0 \le x \le 5$

$f(x,y) = 9x + 8y - 6(x+y)^2$




**Step 1:** Find all possible critical points

$$\frac{\partial f}{\partial x} = 9 - 12(x+y) = 0 \qquad (1) \qquad\qquad \frac{\partial f}{\partial y} = 8 - 12(x+y) = 0 \qquad (2)$$

(1) – (2): $\quad 1 \ne 0 \quad \therefore$ L.H.S. contradicts with R.H.S. Hence, no value for $x$ and $y$ which satisfies both (1) and (2). $\therefore$ **NO CRITICAL POINT**

**Step 2:** Not asked, hence do not need to determine the nature of critical point.

**Step 3.1:** Find the largest and smallest value of $f$ on the boundary $S$.

<u>Edge I</u> $y = 0$ and $x = [0,5]$

$$f(x,y) = 9x + 8y - 6(x+y)^2$$
$$f(x,0) = 9x - 8(0) - 6(x+0)^2 = 9x - 6(x)^2$$

For max./min. $f_x' = 9 - 12x = 0 \Rightarrow x = \frac{3}{4} \Rightarrow$ Critical point $\left(\frac{3}{4}, 0\right)$

<u>Edge II</u> $x = 5$, $0 \le y \le 3$ and $y = [0,3]$

$$f(5,y) = 9(5) + 8y - 6(5+y)^2$$

For max./min. $f_y' = 8 - 12(5+y) = 0 \Rightarrow \cancel{y = -\frac{13}{3} = -4.33}$ ***y* is out of the domain.**

<u>Edge III</u> $y = 3$ and $4 \le x \le 5$

$$f(x,3) = 9x - 8(3) - 6(x+3)^2$$

For max./min. $f_x' = 9 - 12(x+3) = 0 \Rightarrow \cancel{x = -\frac{27}{12} = -2.25}$ ***x* is out of the domain.**

<u>Edge IV</u> $2y - x = 2 \Rightarrow x = 2y - 2$ and $0 \le x \le 4$

$$f((2y-2), y) = 9(2y-2) + 8y - 6(2y - 2 + y)^2$$
$$f((2y-2), y) = 9(2y-2) + 8y - 6(3y-2)^2$$

For max./min. $f_y' = 26 - 12(3y-2)\cdot 3 = 0$

$$\Rightarrow y = \frac{98}{108} = 0.91 \Rightarrow x = 2(0.91)\cancel{-2=-0.18}$$ ***x* is out of the domain.**

<u>Edge V</u> $x = 0$ and $0 \le y \le 1$

$$f(0, y) = 9(0) + 8y - 6(0+y)^2 = 8y - 6y^2$$

For max./min. $f_y' = 8 - 12y = 0 \Rightarrow y = \frac{8}{12} = \frac{2}{3}$ ***y* is in the domain.**

When $y = \frac{2}{3} \Rightarrow$ Critical point $\left(0, \frac{2}{3}\right)$ on the edge V.

**Step 3.2** Compare the value of critical points and points at the end of the boundaries.

$$f(x, y) = 9x + 8y - 6(x+y)^2$$

Critical points
$$\begin{cases} f\left(\frac{3}{4}, 0\right) = 9\left(\frac{3}{4}\right) - 6\left(\frac{3}{4}\right)^2 = \frac{27}{4} - 6\left(\frac{9}{16}\right) = \frac{27}{8} = 3.375 \leftarrow \text{Maximum profit} \\ f\left(0, \frac{2}{3}\right) = 8\left(\frac{2}{3}\right) - 6\left(\frac{2}{3}\right)^2 = \frac{21}{8} = 2.625 \end{cases}$$

Corners $f(0,0) = 0$, $f(5,0) = -105$, $f(5,3) = -315$, $f(4,3) = -234$ and $f(0,1) = 2$

**15. Constraint Optimisation Using Lagrange Multiplier**

Lagrange Multiplier is a method to solve a certain class of **critical points** for multivariable functions. This method will be introduced by the following example:-

**Ex.29:** $Z = f(x, y) = xy$ Find optimum value of $Z$ with the constraint $2x + y = 100$.
There are **TWO** ways to do optimisation with constraints which are

**1. Substitution Method**
**2. Lagrange Multiplier Method**

**1. Substitution Method** (This is possible in simple cases.)

$Z = f(x, y) = xy$ with the constraint $2x + y = 100 \Rightarrow y = 100 - 2x$

$$Z = f(x, 100-2x) = x(100-2x) = 100x - 2x^2$$
$$\frac{\partial f}{\partial x} = 100 - 4x = 0 \Rightarrow x = 25$$

A critical point is $x = 25$ and $y = 100 - 2(25) = 50$.
Hence, $Z$ is optimised when $x = 25$ and $y = 50$.

## 2. Lagrange Multiplier Method

Function $f(x,y)$ with the constraint $g(x,y)=C$. Introduce **Lagrange Function, *L***

$$\boxed{L(x,y)=f(x,y)-\lambda[g(x,y)-C]}$$

$\lambda$ is a constant (known as **Lagrange Multiplier**)

Because $g(x,y)=C$, hence $L(x,y)=f(x,y)$ and maximising *L* is the maximisation of *f* under constraints **OR** optimising *L* is the optimisation of *f* under constraints.

For the optimum value of $L(x,y)$.

Differentiate with respect to *x*: $\quad L_x'(x,y)=f_x'(x,y)-\lambda g_x'(x,y)=0 \quad (1)$

Differentiate with respect to *y*: $\quad L_y'(x,y)=f_y'(x,y)-\lambda g_y'(x,y)=0 \quad (2)$

Constraint equation: $\quad g(x,y)=C \quad (3)$

There are 3 equations and 3 unknowns $(x,y,\lambda)$. Hence, we can solve for *x*, *y* and $\lambda$ which give optimum the value of *L* and hence optimise the value of *f* under constraint.

**Method 1 of solving equations (1) – (3) ➔ Elimination & substitution:**

Function $\quad f(x,y)=xy$

Constraint $\quad 2x+y=100 \quad \{g(x,y)=C\}$

Lagrange function $\quad L(x,y)=f(x,y)-\lambda[g(x,y)-C]$

$\quad L=xy-\lambda(2x+y-100)$

$$\left.\begin{array}{lll} \text{Differentiate with respect to } x\text{:} & L_x'=y-\lambda(2+0-0)=0 \Rightarrow y-2\lambda=0 & (1) \\ \text{Differentiate with respect to } y\text{:} & L_y'=x-\lambda(0+1-0)=0 \Rightarrow x-\lambda=0 & (2) \\ \text{Constraint equation:} & 2x+y=100 & (3) \end{array}\right\} \begin{array}{l}\text{3 equations \&} \\ \text{3 unknowns} \\ (x,y,\lambda)\end{array}$$

(3)-(1): $\quad 2x+2\lambda=100$

$\quad x+\lambda=50 \quad (4)$

(4)+(2): $\quad 2x=50 \Rightarrow x=25$

(2): $\quad \lambda=x=25$

(1): $\quad y=2\lambda=2(25)=50$

Hence, $f(x,y)$ is optimum when *x* = 25 and *y* = 50 (same as substitution method).
The optimum value of the function is $f(25,50)=1250$.

**Method 2 of solving equations (1) – (3) ➔ Matrix:**

$$\begin{array}{rrrcrl} & y & -2\lambda & = & 0 & (1) \\ x & & -\lambda & = & 0 & (2) \\ 2x & +y & & = & 100 & (3) \end{array}$$

System of linear equations

$$\left[\begin{array}{rrr|r} 0 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 100 \end{array}\right] \xrightarrow{\approx} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 100 \end{array}\right]$$

Optimum value $f(25,50)=1250;\ \lambda=25$

**Recap:** Optimise $f(x,y)$ a 2-variable function with the constraint $g(x,y)=C$.
Use Lagrange Multiplier method gets:-

$$L(x,y)=f(x,y)-\lambda[g(x,y)-C]$$

A 2-vaiable function with a constant $\lambda$ but **WITHOUT constraint**.

- Hence, Lagrange Multiplier method can be used to **change constraint multivariable functions to unconstraint multivariable function**.
- Note that this method can **only find critical points**.
- This Lagrange Multiplier method is applicable to any multi-variable functions.

**Ex. 30:** Find critical points of *f* subject to give constraints.

$f(x,y,z)=x^2+xy+2y^2+z^2$ subject to $x-3y-4z=16$

$$L(x,y,z)=f(x,y,z)-\lambda[g(x,y,z)-C]$$

$$\because L=x^2+xy+2y^2+z^2-\lambda\{x-3y-4z-16\}$$

$$\left.\begin{array}{ll} L_x' = 2x+y-\lambda=0 & (1) \\ L_y' = x+4y+3\lambda=0 & (2) \\ L_z' = 2z+4\lambda=0 & (3) \\ x-3y-4z=16 & (4) \end{array}\right\} \text{4 equations 4 unknowns } x,y,z \text{ and } \lambda$$

**Method 1 of solving equations (1) – (4) ➔ Elimination & substitution:**

(2)-(4): $3\lambda+7y+4z=-16$ (5)

(3): $z=-\dfrac{4\lambda}{2} \Rightarrow z=-2\lambda$ (3a)

Substitute $z$ into (5): $3\lambda+7y-8\lambda=-16$

$7y-5\lambda=-16 \Rightarrow y=\dfrac{5\lambda-16}{7}$ (6)

Rearrange (4): $x=3y+4z+16$ (4a)

Substitute (4a) into (1): $2(3y+4z+16)+y-\lambda=0$

$6y+8z+32+y-\lambda=0$

$7y+8z+32-\lambda=0$ (1a)

From (3a), (6), (1a) becomes $7\left(\dfrac{5\lambda-16}{7}\right)+8(-2\lambda)+32-\lambda=0$

$5\lambda-16-16\lambda+32-\lambda=0$

$-12\lambda+16=0 \Rightarrow \lambda=\dfrac{16}{12}=\dfrac{4}{3}$

From (3a): $z=-2\left(\frac{4}{3}\right)=-\frac{8}{3}$

From (6): $y=\dfrac{5\left(\frac{4}{3}\right)-16}{7}=\frac{1}{7}\cdot\left(\frac{20-48}{3}\right)=-\frac{28}{21}=-\frac{4}{3}$

From (4): $x=3\left(-\frac{4}{3}\right)+4\left(-\frac{8}{3}\right)+16=\frac{48-12-32}{3}=\frac{4}{3}$

Hence, critical point is $\left(\frac{4}{3},-\frac{4}{3},-\frac{8}{3}\right)$.

**Method 2 of solving equations (1) – (4) ➔ Matrix:**

$$\left.\begin{array}{lllllr} 2x & +y & & -\lambda & =0 & (1)\\ x & +4y & & +3\lambda & =0 & (2)\\ & & 2z & +4\lambda & =0 & (3)\\ x & -3y & +4z & & =16 & (4) \end{array}\right.$$ System of linear equations

$(x,y,z)=\left(\frac{4}{3},-\frac{4}{3},-\frac{8}{3}\right)$ and $\lambda=\frac{4}{3}$

**Number of Lagrange Multiplier = Number of constraint equations.**

**Ex. 31:** Optimise $f(x,y,z)=x^2+y^2+z^2$ subject to $x+y+z=4$ and $x-y+z=4$

$$L(x,y,z)=f(x,y,z)-\lambda_1[g_1(x,y,z)-C_1]-\lambda_2[g_2(x,y,z)-C_2]$$

$$L=x^2+y^2+z^2-\lambda_1(x+y+z-4)-\lambda_2(x-y+z-4)$$

$$\left.\begin{array}{ll} L_x'=2x-\lambda_1-\lambda_2=0 & (1)\\ L_y'=2y-\lambda_1+\lambda_2=0 & (2)\\ L_z'=2z-\lambda_1-\lambda_2=0 & (3)\\ x+y+z=4 & (4)\\ x-y+z=4 & (5) \end{array}\right\}$$ 5 equations 5 unknowns $x,y,z,\lambda_1$ and $\lambda_2$

**Method 1:**
(4)-(5): $2y=0 \quad\Rightarrow\quad y=0$
(1)-(3): $2x-2z=0 \quad\Rightarrow\quad x=z$
(4): $x+0+x=4 \quad\Rightarrow\quad x=2=z$
Hence, critical pint is $(2,0,2)$

**Method 2:**

$$\begin{array}{llllllll} 2x & & & -\lambda_1 & -\lambda_2 & =0 & (1) & \\ & 2y & & -\lambda_1 & +\lambda_2 & =0 & (2) & \\ & & 2z & -\lambda_1 & -\lambda_2 & =0 & (3) & \text{System of linear equations} \\ x & +y & +z & & & =4 & (4) & \\ x & -y & +z & & & =4 & (5) & \end{array}$$

$(x,y,z)=(2,0,2)$ and $\lambda_1=\lambda_2=2$

**Ex. 32:** A firm production function is $f(l,k)=12l+20k-l^2-2k^2$. Material costs are $l$ costs 4 per unit and $k$ costs 8 per unit. The budget allows the total cost to be 88. Find the maximum output that this firm can produce under this budget constraint. $\Rightarrow 4l+8k=88$

$$L(x,y)=f(x,y)-\lambda[g(x,y)-C]$$
$$L=12l+20k-l^2-2k^2-\lambda(4l+8k-88)$$

$$\left.\begin{array}{ll} L_l'=12-2l-4\lambda=0 & (1) \\ L_k'=20-4k-8\lambda=0 & (2) \\ 4l+8k=88 & (3) \end{array}\right\} \begin{array}{l} \text{Solve 3 equations 3 unknowns} \\ \text{to get } l=8 \text{ and } k=7 \\ \therefore\ f(8,7)=74 \end{array}$$

If use matrix to solve equations (1) – (3);

$$\left.\begin{array}{lllll} 2l & & +4\lambda & =12 & (1) \\ & 4k & +8\lambda & =20 & (2) \\ 4l & +8K & & =88 & (3) \end{array}\right\} (l,k,\lambda)=(8,7,-1) \text{ and } \therefore\ f(8,7)=74$$

**Ex. 33:** A utility function $U(x,y)=Ax^a y^b$ and a budget constraint $px+qy=m$ where $A,a,b,p,q$ and $m$ are positive constants, $x$ is the price and $y$ is the income. Find the price and the income that optimum the customer demand.

$$L=Ax^a y^b-\lambda(px+qy-m)$$
$$L_x'=Aax^{a-1}y^b-\lambda p=0 \qquad (1)$$
$$L_y'=Abx^a y^{b-1}-\lambda q=0 \qquad (2)$$
$$px+qy=m \qquad (3)$$

(1): $\lambda p=Aax^{a-1}y^b \quad \Rightarrow \quad \lambda=\dfrac{Aax^{a-1}y^b}{p}$ (1a)

(2): $\lambda q=Abx^a y^{b-1} \quad \Rightarrow \quad \lambda=\dfrac{Abx^a y^{b-1}}{q}$ (2a)

(1a)=(2a): $\dfrac{Aax^{a-1}y^b}{p}=\dfrac{Abx^a y^{b-1}}{q}$

$$\frac{ay}{p} = \frac{bx}{q} \quad \Rightarrow \quad qy = \frac{bxp}{a} \qquad (4)$$

Substitute (4) into (3): $\quad px + \dfrac{bxp}{a} = m$

$$px\left(1 + \frac{b}{a}\right) = m$$

$$px\left(\frac{a+b}{a}\right) = m$$

Similarly,

$$\boxed{\begin{aligned} x(p,q,m) = x &= \left(\frac{a}{a+b}\right)\frac{m}{p} \\ y(p,q,m) = y &= \left(\frac{b}{a+b}\right)\frac{m}{q} \end{aligned}}$$

Note that if both price and income increase by a factor $t$ then

$$x(tp,tq,tm) = \left(\frac{a}{a+b}\right)\frac{tm}{tp} = x(p,q,m)$$

The values of $a$ and $b$ are the important factors for $x$ and $y$.

## 16. Economic meaning of Lagrange Multiplier

Lagrange function of $f(x,y)$ with constraint $g(x,y) = C$ is

$$\boxed{L(x,y) = f(x,y) - \lambda[g(x,y) - C]} \qquad (1)$$

To solve for the optimum $x$ and $y$

$$\boxed{\begin{aligned} &L_x'(x,y) = f_x'(x,y) - \lambda g_x'(x,y) = 0 \\ &L_y'(x,y) = f_y'(x,y) - \lambda g_y'(x,y) = 0 \\ &g(x,y) = C \end{aligned}} \qquad \begin{aligned} (2) \\ (3) \\ (4) \end{aligned}$$

Let $x^*$ and $y^*$ be the values of $x$ and $y$ which give optimum $f(x,y)$. $x^*$ and $y^*$ depend on $C$. Hence, this can be written as

$$x^* = x^*(C) \qquad \text{and} \qquad y^* = y^*(C)$$

and $\quad f^*(C) = f^*(x^*(C), y^*(C))$ the optimum value of function

**Chain Rule** $\quad \dfrac{\partial f^*(C)}{\partial C} = f_{x^*}'(x^*, y^*) \cdot \dfrac{\partial x^*}{\partial C} + f_{y^*}'(x^*, y^*) \cdot \dfrac{\partial y^*}{\partial C} \qquad (4)$

Consider $f_{x^*}'(x^*, y^*)$ and $f_{y^*}'(x^*, y^*)$:

From (2) and (3): $f_x'(x,y) = \lambda g_x'(x,y)$ and $f_y'(x,y) = \lambda g_y'(x,y)$

$\therefore$ (4) becomes $\dfrac{\partial f^*(C)}{\partial C} = \lambda g_x'(x^*,y^*)\cdot\dfrac{\partial x^*}{\partial C} + \lambda g_y'(x^*,y^*)\cdot\dfrac{\partial y^*}{\partial C}$

$$\boxed{\frac{\partial f^*(C)}{\partial C} = \lambda\left[g_x'(x^*,y^*)\cdot\frac{\partial x^*}{\partial C} + g_y'(x^*,y^*)\cdot\frac{\partial y^*}{\partial C}\right]} \qquad (5)$$

Consider the constraint equation $g(x,y) = C$, also $g(x^*,y^*) = C$

Partial differentiate with respect to $C$ on both sides of the equation.

$$\frac{\partial g(x^*,y^*)}{\partial C} = \frac{\partial C}{\partial C}$$

Chain rule: $g_x'(x^*,y^*)\cdot\dfrac{\partial x^*}{\partial C} + g_y'(x^*,y^*)\cdot\dfrac{\partial y^*}{\partial C} = 1$

$\therefore$ From (5): $\dfrac{\partial f^*(C)}{\partial C} = \lambda[1]$

$$\boxed{\frac{\partial f^*(x^*,y^*)}{\partial C} = \lambda} \qquad \textbf{Lagrange Multiplier}$$

$f(x,y)$ is utility or profit function.
$C$ is the resources e.g. budget $g(x,y) = C$.
$\lambda$ is the rate of change of utility or profit per resource (budget) or the shadow price of the resource.

**Ex. 34a:** A production function $Q = 120kl$ with the constraint $2k + 5l = m$ where $Q$ is the number of units produced, $k$ is the capital, $m$ is the budget and $l$ is the number of labour. Estimate how many more units are produced if $m$ is increased by 1 unit from 100 to 101.

From the last example, $U(x,y) = Ax^a y^b$ subject to $pk + ql = m$

$$x^* = \left(\frac{a}{a+b}\right)\frac{m}{p} \quad \text{and} \quad y^* = \left(\frac{b}{a+b}\right)\frac{m}{q}$$

$\because x = k, \quad y = l, \quad A = 120, \quad a = 1, \quad b = 1, \quad p = 2, \quad q = 5$

$$\therefore \quad k^* = \left(\frac{1}{1+1}\right)\frac{m}{2} = \frac{m}{4} \qquad\qquad l^* = \left(\frac{1}{1+1}\right)\frac{m}{5} = \frac{m}{10}$$

The optimum unit produced is $Q^* = 120k^*l^* = 120\left(\dfrac{m}{4}\right)\left(\dfrac{m}{10}\right) = 3m^2$

$$\lambda = \frac{\partial Q^*}{\partial m} = 6m$$

**Proof** $L = 120kl - \lambda(2k + 5l - m)$

$L_k' = 120l - 2\lambda = 0$

$$120\left(\frac{m}{10}\right) - 2\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{12m}{m} = 6m$$

Consider when $m$ = 100, $k^* = 25$, $l^* = 10$ and $\lambda = 600$

$Q^* = 120 \cdot 25 \cdot 10 = 30{,}000$

What will happen if the budget is increased by 1 unit from 100 to 101?

New $k^* = \dfrac{101}{4} = 25.25$ New $l^* = \dfrac{101}{10} = 10.1$ New $Q^* = 3(101)^2 = 30603$

Hence, $Q$ is increased by 30,603-30,000 = 603 $\approx \lambda$.

**Ex. 34b:** For a three variable function

$$f(x, y, z) = 2\ln\left(\frac{1}{x}\right) + 3\ln\left(\frac{1}{y}\right) + \ln\left(\frac{1}{z}\right)$$

subject to the constraint

$$3x + 2y - \frac{1}{z} = 1$$

(a) Use **the Lagrange multiplier method** to find all critical point(s) of the function and the corresponding minimum value(s) of the function $f(x, y, z)$.

(b) **Estimate** the new optimum value(s) of the function $f(x, y, z)$ if the constraint is changed to $3x + 2y - \dfrac{1}{z} = 1.1$.

[Ans: $f\left(\frac{1}{6}, \frac{3}{8}, 4\right) = 5.14,\ f_{new} = 4.74$]

**Ex. 34c:** For a three-variable function

$$f(x, y, z) = xyz$$

subject to the constraints

$$x + y + z = 12 \text{ and } x + y - z = 0 \quad (xyz \neq 0),$$

(a) **Use the Lagrange multiplier method** to find all the critical point(s) of this functio and the corresponding optimum value(s) of the function $f(x, y, z)$.

(7 marks)

(b) **Estimate** the new optimum value(s) of the function $f(x, y, z)$ if the constraints are changed to $x + y + z = 12$ and $x + y - z = 0.2$? (2 marks)

## 17. Optimisation with Inequality Constraints

Inequality constraints $h(x,y) \le b$ e.g. the production cost must not exceed 10,000 $(\le 10{,}000)$ or simply price cannot be negative $(p \ge 0)$.

***Using Lagrange Multiplier Method***

Find the optimum $f(x,y)$ subject to $h(x,y) \le b$.

The Lagrange function $\boxed{L(x,y) = f(x,y) - \mu[h(x,y) - b]}$

Conditions for optimum $x$, $y$ and $\mu$

| | |
|---|---|
| Differentiate with respect to $x$: | $L_x'(x,y) = f_x'(x,y) - \mu\, h_x'(x,y) = 0$ |
| Differentiate with respect to $y$: | $L_y'(x,y) = f_y'(x,y) - \mu\, h_y'(x,y) = 0$ |
| Constraint: | $h(x,y) \le b$ |
| Complementary slackness conditions | $\mu \ge 0$ |
| and | $\mu[h(x,y) - b] = 0$ |

The above five equations are known as **the Kuhn-Tucker conditions** for the solution of optimisation problems with inequality constraints.

Similarly form the economic interpretation:

$$\mu = \frac{\partial f(x^*, y^*)}{\partial C}$$

**Ex. 35:** Maximise $f(x,y) = x^2 + y^2 + y - 1$ subject to $x^2 + y^2 \le 1$.

$$L = x^2 + y^2 + y - 1 - \mu\left(x^2 + y^2 - 1\right)$$

$$\left.\begin{array}{ll} L_x' = 2x - 2\mu x = 0 & (1) \\ L_y' = 2y + 1 - 2\mu y = 0 & (2) \\ x^2 + y^2 \le 1 & (3) \\ \mu \ge 0 & (4) \\ \mu\left(x^2 + y^2 - 1\right) = 0 & (5) \end{array}\right\} \text{Need to find } x, y \text{ and } \mu \text{ that satisfy all 5 equations.}$$

(1): $2x(1-\mu) = 0$

Hence, $x = 0$ or $(1-\mu) = 0 \Rightarrow \mu = 1$

(2): $\mu = 1$ $\quad 2y + 1 - 2(1)y = 0$

$\quad 1 \ne 0 \quad \therefore \mu \ne 1$

$\therefore \quad x = 0$

(3): Maximum point can occur when **Case 1** $x^2 + y^2 = 1$ or **Case 2** $x^2 + y^2 < 1$

**Case 1**: $x^2 + y^2 = 1$ {satisfy (5)}, if $x = 0$, then $y = \pm 1$

**Case 1.1**: If $y$ = 1, then (2): $2(1) + 1 - 2\mu(1) = 0$

$$2\mu = 3 \quad \Rightarrow \quad \mu = \frac{3}{2} \geq 0 \qquad \text{Satisfy (4)}$$

$\therefore \quad x = 0, y = 1$ and $\mu = \frac{3}{2}$ is a critical point.

**Case 1.2**: If $y$ = -1, then (2): $2(-1) + 1 - 2\mu(-1) = 0$

$$2\mu = 1 \quad \Rightarrow \quad \mu = \frac{1}{2} \geq 0 \qquad \text{Satisfy (4)}$$

$\boxed{\therefore \quad x = 0, y = -1 \text{ and } \mu = \frac{1}{2} \text{ is a critical point.}}$

**Case 2**: $x^2 + y^2 < 1$, (5): $\mu(x^2 + y^2 - 1) = 0$ and $(x^2 + y^2 - 1) \neq 0$, $\therefore \quad \mu = 0$ Satisfy (4) and (5)

$x = 0$, then $y^2 < 1 \quad \Rightarrow \quad -1 < y < 1$

(2): $2y + 1 - 2(0)y = 0 \quad \Rightarrow \quad y = -\frac{1}{2}$

$\boxed{\therefore \quad x = 0, y = -\frac{1}{2} \text{ and } \mu = 0 \text{ is a critical point.}}$

$$\left.\begin{aligned} f(0,1) &= 0^2 + 1^2 + 1 - 1 = 1 \\ f(0,-1) &= 0^2 + (-1)^2 + (-1) - 1 = -1 \\ f\left(0, -\frac{1}{2}\right) &= 0^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 1 = -\frac{5}{4} \end{aligned}\right\} \text{Maximum } f \text{ is when } x = 0, y = 1 \text{ and } \mu = \frac{3}{2}.$$

**Ex. 36:** Maximise $f(x, y) = x - \frac{x^2}{2} + y^2$ subject to $\frac{x^2}{2} + y^2 \leq \frac{9}{8}$ and $-y \leq 0$.

$$L = x - \frac{x^2}{2} + y^2 - \mu_1\left(\frac{x^2}{2} + y^2 - \frac{9}{8}\right) - \mu_2(-y - 0)$$

| | | | |
|---|---|---|---|
| $L_x' = 1 - x - \mu_1 x = 0$ | (1) | $L_y' = 2y - 2\mu_1 y + \mu_2 = 0$ | (2) |
| $\frac{x^2}{2} + y^2 \leq \frac{9}{8}$ | (3) | $-y \leq 0$ | (4) |
| $\mu_1 \geq 0$ | (5) | $\mu_1\left(\frac{x^2}{2} + y^2 - \frac{9}{8}\right) = 0$ | (6) |
| $\mu_2 \geq 0$ | (7) | $\mu_2(-y) = 0$ | (8) |

From constraint equations (3) and (4) and complimentary slackness conditions (5)-(8), possible solutions can be when

| Equations | (3) | (4) | (5) & (6) | (7) & (8) |
|---|---|---|---|---|
| **Case 1** | $\frac{x^2}{2}+y^2=\frac{9}{8}$ | $-y=0$ | | |
| **Case 2** | $\frac{x^2}{2}+y^2<\frac{9}{8}$ | $-y<0$ | $\mu_1=0$ | $\mu_2=0$ |
| **Case 3** | $\frac{x^2}{2}+y^2=\frac{9}{8}$ | $-y<0$ | | $\mu_2=0$ |
| **Case 4** | $\frac{x^2}{2}+y^2<\frac{9}{8}$ | $-y=0$ | $\mu_1=0$ | |

**Case 1**: $\frac{x^2}{2}+y^2=\frac{9}{8}$ (3a) and $-y=0$ (4a) Satisfy (3) and (4)

(4a): $y$ = 0, then (8) is satisfied. { $\mu_2(-y)=0$ }

(2): $2(0)-2\mu_1(0)+\mu_2=0 \quad\Rightarrow\quad \mu_2=0$

(3a): $\frac{x^2}{2}+0^2=\frac{9}{8} \quad\Rightarrow\quad x=\pm\frac{3}{2}$

**Case 1.1**: $x=\frac{3}{2}$

(1): $1-\left(\frac{3}{2}\right)-\mu_1\left(\frac{3}{2}\right)=0 \quad\Rightarrow\quad \cancel{\mu_1=-\frac{1}{3}}$ Not satisfy (5)

**Case 1.2**: $x=-\frac{3}{2}$

(1): $1-\left(-\frac{3}{2}\right)-\mu_1\left(-\frac{3}{2}\right)=0 \quad\Rightarrow\quad \cancel{\mu_1=-\frac{5}{3}}$ Not satisfy (5)

**Case 2**: $-y<0$, $\mu_1=0$ and $\mu_2=0$ Satisfy (4) – (8)

(1): $1-x-(0)x=0 \quad\Rightarrow\quad x=1$ Satisfy (1)

(2): $2y-2(0)y+(0)=0 \quad\Rightarrow\quad y=0$ Satisfy (2)

$y=0$ is out of range since $-y<0$. There is no critical point in Case 2

**Case 3**: $\frac{x^2}{2}+y^2=\frac{9}{8}$ (3a) and $\mu_2=0$ Satisfy (6) - (8)

(2): $2y-2\mu_1 y+(0)=0 \quad\Rightarrow\quad \mu_1=\frac{2y}{2y}=1$ Satisfy (2) and (5)

(1): $1-x-(1)x=0 \quad\Rightarrow\quad 2x=1 \quad\Rightarrow\quad x=\frac{1}{2}$ Satisfy (1)

(3a): $\frac{\left(\frac{1}{2}\right)^2}{2}+y^2=\frac{9}{8} \quad\Rightarrow\quad y^2=\frac{9}{8}-\frac{1}{8}=1 \quad\Rightarrow\quad y=\pm1$ Satisfy (3)

(4): $-y\le 0 \quad\Rightarrow\quad y\ge 0$ Hence, only $y$ = 1 Satisfy (4)

$$\boxed{\therefore\quad x=\frac{1}{2} \text{ and } y=1 \text{ is a critical point.}}$$

**Case 4**: $-y=0$ and $\mu_1=0$ Satisfy (4) – (6)

(1): $1-x-(0)x=0 \quad \Rightarrow \quad x=1$ Satisfy (1)

(2): $2y-2(0)y+(0)=0 \quad \Rightarrow \quad y=0$ Satisfy (2)

Check (3): $\frac{x^2}{2}+y^2=\frac{1^2}{2}+0^2=\frac{1}{2}\le\frac{9}{8}$ Satisfy (3)

$$\boxed{\therefore \quad x=1 \text{ and } y=0 \text{ is a critical point.}}$$

$$\left.\begin{aligned} f(1,0)&=1-\frac{1^2}{2}+0^2=\frac{1}{2} \\ f\left(\frac{1}{2},1\right)&=\left(\frac{1}{2}\right)-\frac{\left(\frac{1}{2}\right)^2}{2}+1^2=\frac{1}{2}-\frac{1}{8}+1=\frac{11}{8}\end{aligned}\right\} \qquad \text{Maximum } f \text{ is when } x=\frac{1}{2} \text{ and } y=1.$$

**Ex. 37:** Maximise $f(x,y)=xy$ subject to $x^2+y^2\le 1$.

$$L=xy-\mu\left(x^2+y^2-1\right)$$

$$L_x'=y-2\mu x=0 \qquad (1)$$
$$L_y'=x-2\mu y=0 \qquad (2)$$
$$x^2+y^2\le 1 \qquad (3)$$
$$\mu\ge 0 \qquad (4)$$
$$\mu\left(x^2+y^2-1\right)=0 \qquad (5)$$

$$\left.\begin{aligned} (1):&\quad \mu=\frac{y}{2x} \quad (1a) \\ (2):&\quad \mu=\frac{x}{2y} \quad (2a)\end{aligned}\right\} \qquad \frac{y}{2x}=\frac{x}{2y} \quad \Rightarrow \quad x^2=y^2 \qquad (6)$$

**Case 1**: $x^2+y^2=1$ (3a) Satisfy (3) and (5)

(3a) and (6): $x^2+x^2=1 \quad \Rightarrow \quad 2x^2=1 \quad \Rightarrow \quad x^2=\frac{1}{2} \quad \Rightarrow \quad x=\pm\sqrt{\frac{1}{2}}$ and $y=\pm\sqrt{\frac{1}{2}}$

| | $x$ | $y$ | $\mu = \dfrac{y}{2x}$ (1a) |
|---|---|---|---|
| Case 1.1 | $x = \sqrt{\dfrac{1}{2}}$ | $y = \sqrt{\dfrac{1}{2}}$ | $\mu = \dfrac{\sqrt{\frac{1}{2}}}{2\sqrt{\frac{1}{2}}} = \dfrac{1}{2} \geq 0$ Satisfy (4) |
| Case 1.2 | $x = \sqrt{\dfrac{1}{2}}$ | $y = -\sqrt{\dfrac{1}{2}}$ | $\mu = \dfrac{-\sqrt{\frac{1}{2}}}{2\sqrt{\frac{1}{2}}} = -\dfrac{1}{2} \leq 0$ Not satisfy (4) |
| Case 1.3 | $x = -\sqrt{\dfrac{1}{2}}$ | $y = \sqrt{\dfrac{1}{2}}$ | $\mu = \dfrac{\sqrt{\frac{1}{2}}}{2\left(-\sqrt{\frac{1}{2}}\right)} = -\dfrac{1}{2} \leq 0$ Not satisfy (4) |
| Case 1.4 | $x = -\sqrt{\dfrac{1}{2}}$ | $y = -\sqrt{\dfrac{1}{2}}$ | $\mu = \dfrac{-\sqrt{\frac{1}{2}}}{2\left(-\sqrt{\frac{1}{2}}\right)} = \dfrac{1}{2} \geq 0$ Satisfy (4) |

Critical points are $\left(\sqrt{\dfrac{1}{2}}, \sqrt{\dfrac{1}{2}}\right)$ and $\left(-\sqrt{\dfrac{1}{2}}, -\sqrt{\dfrac{1}{2}}\right)$.

**Case 2**: $x^2 + y^2 < 1$ and from (4) $\mu = 0$ Satisfy (3) – (5)

(1): $y - 2(0)x = 0 \Rightarrow y = 0$ Satisfy (1)

(2): $x - 2(0)y = 0 \Rightarrow x = 0$ Satisfy (2)

A critical point is $(0,0)$.

$$\left.\begin{aligned} &f(0,0) = 0 \cdot 0 = 0 \\ &f\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = \sqrt{\tfrac{1}{2}} \cdot \sqrt{\tfrac{1}{2}} = \tfrac{1}{2} \\ &f\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) = \left(-\sqrt{\tfrac{1}{2}}\right) \cdot \left(-\sqrt{\tfrac{1}{2}}\right) = \tfrac{1}{2} \end{aligned}\right\}$$

Maximum $f(x, y) = xy$ is when $\left(\sqrt{\dfrac{1}{2}}, \sqrt{\dfrac{1}{2}}\right)$ and $\left(-\sqrt{\dfrac{1}{2}}, -\sqrt{\dfrac{1}{2}}\right)$.

## 18. The Envelope Theorem

The Envelope theorem describes how the optimal value of the objective function in a parameterised optimisation problem changes as one of the parameters changes. The envelope theorem is useful to determine the effect of parametric changes without actually solving for the optimal output. This is an approximation but fast.

An objective function $f(x,r)$

To maximise $f(x,r)$ with respect to $x$ while $r$ is constant,

$\text{Max}_x\ f(x,r)$

$x$ that maximises $f$ at a constant $r$ is $x^*(r)$

So $f(x,r) \rightarrow f\left(x^*(r),r\right)$ Value function

When $r$ changes, what will happen? To answer this, we need to know $\dfrac{df\left(x^*(r),r\right)}{dr}$.

Chain rule: $$\frac{df\left(x^*(r),r\right)}{dr} = \underbrace{\frac{\partial f\left(x^*(r),r\right)}{\partial x}\cdot\frac{\partial x}{\partial r}}_{(1)} + \underbrace{\frac{\partial f\left(x^*(r),r\right)}{\partial r}\boxed{\frac{\partial r}{\partial r}}}_{(2)} \qquad \boxed{\frac{\partial r}{\partial r}} = 1$$

$x^*(r)$ maximises $f$, the first derivative test $\dfrac{\partial f\left(x^*(r),r\right)}{\partial x} = 0$

$\therefore$ (1): $$\frac{\partial f\left(x^*(r),r\right)}{\partial x}\cdot\frac{\partial x}{\partial r} = 0$$

$$\therefore \quad \frac{df\left(x^*(r),r\right)}{dr} = \frac{\partial f\left(x^*(r),r\right)}{\partial r}$$

**Envelope Theory** $$\therefore \quad \boxed{\frac{df\left(x^*(r),r\right)}{dr} = \frac{\partial f\left(x^*(r),r\right)}{\partial r} = f_r{}'\left(x^*(r),r\right)}$$

**Ex. 38:** $f(x) = 4x^2 + 16xa + 2a^2$ Estimate the change of the optimum values of $f$ when $a$ changes from 1 to 1.1.

There are several methods of solving this.

**Method 1**: Calculate the optimum value of $f(x)$ for $a$ = 1 and $a$ = 1.1 and then calculate the change in the optimum value of $f(x)$.

$a$ = 1 $\quad f(x) = 4x^2 + 16x + 2$

$$\frac{df}{dx} = 8x + 16 = 0 \quad \Rightarrow \quad x^* = -2$$

$$\frac{d^2 f}{dx^2} = 8 > 0 \qquad \therefore \quad x^* = -2 \text{ is a relative minimum point}$$

$$f(-2) = 4(-2)^2 + 16(-2) + 2 = -14$$

$a$ = 1.1 $\quad f(x) = 4x^2 + 16x(1.1) + 2(1.1)^2$

$$f(x) = 4x^2 + 17.6x + 2.42$$

$$\frac{df}{dx} = 8x + 17.6 = 0 \quad \Rightarrow \quad x^* = -\frac{17.6}{8} = -2.2$$

$$\frac{d^2 f}{dx^2} = 8 > 0 \qquad \therefore \quad x^* = -2.2 \text{ is a relative minimum point}$$

$$f(-2.2) = 4(-2.2)^2 + 17.6(-2.2) + 2.42 = -16.94$$

Hence, the optimum value changes by -16.94 – (-14) = -2.94 ← Exact change

**Method 2**: Envelope theory

$$f(x) = 4x^2 + 16xa + 2a^2$$

$$\frac{\partial f}{\partial x} = 8x + 16a = 0 \quad \Rightarrow \quad x^* = \frac{-16a}{8} = -2a$$

$$\frac{\partial f}{\partial a} = 16x + 4a$$

$$\frac{\partial f^*}{\partial a} = 16(-2a) + 4a = -28a$$

Envelope theory

$$\frac{df(x^*, a)}{da} = \frac{\partial f(x^*, a)}{\partial a} = -28a \quad \Rightarrow \quad df^*(x^*, a) = (-28a)\, da$$

$a_1 = 1$, $a_2 = 1.1$, $\Delta a = a_2 - a_1 = 1.1 - 1 = 0.1$

$\therefore \quad df^*(x^*, a) = (-28) \cdot (1) \cdot (0.1) = -2.8$ ← Estimated change

**Ex. 39a:** $f(x,a) = -x^2 + 2ax + 4a^2$ What is the approximate effect of a unit increase in *a* on a maximum value of *f*? (Use Envelope theorem)

$$\frac{\partial f}{\partial x} = -2x + 2a = 0 \quad \Rightarrow \quad x^* = a$$

$$f(x^*(a), a) = -a^2 + 2a(a) + 4a^2 = 5a^2$$

$$\frac{\partial f(x^*(a), a)}{\partial a} = 10a = \frac{df(x^*(a), a)}{da} \qquad \text{Envelop theory}$$

$$df(x^*(a), a) = 10a\, da$$

If $da = 1 \quad \Rightarrow \quad df = 10a$

$\therefore$ As *a* increases by 1 unit, the optimum value of *f* increase by 10*a* approximately.

**Ex. 39b:** A company sells 2 products *A* and *B* with the demand functions as follows:

$$q_A = 50 - \frac{1}{2} p_A$$
$$q_B = 76 - p_B$$

The **total cost** is $\quad c = 3q_A^{\ 2} + 2q_A q_B + 2q_B^{\ 2} + 55$

(a) Calculate the optimum profit and use the **second derivative test** to show that this optimum maximises profit.

(b) If the accountant made a mistake in the calculation of the total cost, the **new total cost** becomes $\quad c = 3q_A^{\ 2} + 2q_A q_B + 1.8q_B^{\ 2} + 55$,

apply the **envelop theorem** to **approximate** the new maximum profit.

[Ans: $q_A = 8, q_B = 10, P_{\max} = 725, P_{new} = 741$]

**Ex. 39c:** Use Lagrange Multiplier method to solve

$$\max \quad f(x, y) = x + 2y \quad \text{subject to} \quad x^2 + y^2 = 1 \quad \text{and} \quad -x - y \le 0$$

Use the envelope theorem to estimate the maximum value of $f(x, y) = x + 4y$ subject to the same constraints.

[Ans: max $f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{5}{\sqrt{5}}, f_{new} = \frac{9}{\sqrt{5}}$ ]

**Ex. 40:** A company produces *q* units of product, sold at *p* baht per piece. Cost of production is *C*. Unfortunately a fraction $1-\alpha$ of products are defective and cannot be sold. Find how the improvement in product quality affects the company's profit.

If *q* units are good, the profit is $\quad P_o = R - C = pq - Cq$

If $(1-\alpha)q$ units are defected, the profit is $\quad P_n = p\{1-(1-\alpha)\}q - Cq$

$$P_n = p\alpha q - Cq$$
$$\frac{\partial P_n}{\partial \alpha} = pq \approx \frac{dP_n}{d\alpha}$$

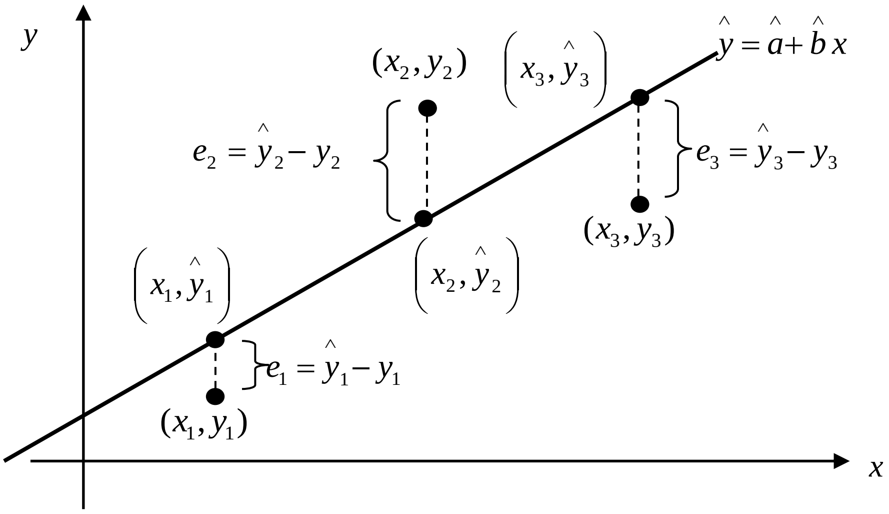
Since *p* and *q* are always positive. Hence, $\frac{dP_n}{d\alpha} > 0$.

$\alpha$ is the fraction of good products. If $d\alpha$ increases, $dP_n$ will increases.
Hence, the firm will have more profits if the quantity of the products sold increases.

**Ex. 41:** A profit of a firm is $p(q) = R(q) - C(q) - tq$ where *t* is the tax per unit sold. What happens to $q^*$ and $p(q^*)$ if the tax varies. $q^*$ gives maximise profit.

[Ans: $\frac{dp}{dt} = -q^* \qquad \therefore \quad t\uparrow \quad p\downarrow$ by $q^*$]

## 19. Line of Regression (Least Squares Approximation)




Line of regression is the best fit linear line for a group of data. The summation of the error is $E = e_1 + e_2 + e_3$. The error can cancel each other out so this is not a good representative.

On the other hand, $S = e_1^2 + e_2^2 + e_3^2$ can be used. The best fit linear line occurs when $S$ is minimised.

$$S = \left(\hat{y}_1 - y_1\right)^2 + \left(\hat{y}_2 - y_2\right)^2 + \left(\hat{y}_3 - y_3\right)^2$$

Since, $\hat{y} = \hat{a} + \hat{b}x$

Hence, $S = \left(\hat{a} + \hat{b}x_1 - y_1\right)^2 + \left(\hat{a} + \hat{b}x_2 - y_2\right)^2 + \left(\hat{a} + \hat{b}x_3 - y_3\right)^2$

Generalise to $n$ data points,

$$S\left(\hat{a}, \hat{b}\right) = \left(\hat{a} + \hat{b}x_1 - y_1\right)^2 + \left(\hat{a} + \hat{b}x_2 - y_2\right)^2 + \ldots + \left(\hat{a} + \hat{b}x_n - y_n\right)^2$$

To minimise $S$, $\dfrac{\partial S}{\partial \hat{a}} = 0$ and $\dfrac{\partial S}{\partial \hat{b}} = 0$

$$\frac{\partial S}{\partial \hat{a}} = 2\left(\hat{a} + \hat{b}x_1 - y_1\right) + 2\left(\hat{a} + \hat{b}x_2 - y_2\right) + \ldots + 2\left(\hat{a} + \hat{b}x_n - y_n\right) = 0 \qquad (1)$$

$$\frac{\partial S}{\partial \hat{b}} = 2\left(\hat{a} + \hat{b}x_1 - y_1\right)\cdot x_1 + 2\left(\hat{a} + \hat{b}x_2 - y_2\right)\cdot x_2 + \ldots + 2\left(\hat{a} + \hat{b}x_n - y_n\right)\cdot x_n = 0 \qquad (2)$$

(1): $\quad 2\hat{a}n + 2\hat{b}\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} y_i = 0 \qquad$ (1a)

(2): $\quad 2\hat{a}\sum_{i=1}^{n} x_i + 2\hat{b}\sum_{i=1}^{n} x_i^{\,2} - 2\sum_{i=1}^{n} x_i y_i = 0 \qquad$ (2a)

For simplification, write $\sum_{i=1}^{n} x_i = \sum_n x_i$ , hence

(1a): $\hat{a}n + \hat{b}\sum_n x_i - \sum_n y_i = 0$ (1b)

(2a): $\hat{a}\sum_n x_i + \hat{b}\sum_n x_i^2 - \sum_n x_i y_i = 0$ (2b)

2 equations and 2 unknowns

Solve for $\hat{a}$ and $\hat{b}$

Solve for $\hat{b}$ first (eliminate $\hat{a}$),

(1b)$\times\sum_n x_i$ : $\hat{a}n\sum_n x_i + \hat{b}\left(\sum_n x_i\right)^2 - \sum_n x_i \sum_n y_i = 0$ (3)

(2b) $\times n$: $\hat{a}n\sum_n x_i + \hat{b}n\sum_n x_i^2 - n\sum_n x_i y_i = 0$ (4)

(3)-(4): $\hat{b}\left(\sum_n x_i\right)^2 - \sum_n x_i \sum_n y_i - \hat{b}n\sum_n x_i^2 + n\sum_n x_i y_i = 0$

$$\hat{b}\left\{\left(\sum_n x_i\right)^2 - n\sum_n x_i^2\right\} = \sum_n x_i \sum_n y_i - n\sum_n x_i y_i$$

$$\boxed{\hat{b} = \frac{\sum_n x_i \sum_n y_i - n\sum_n x_i y_i}{\left(\sum_n x_i\right)^2 - n\sum_n x_i^2}}$$

And hence,

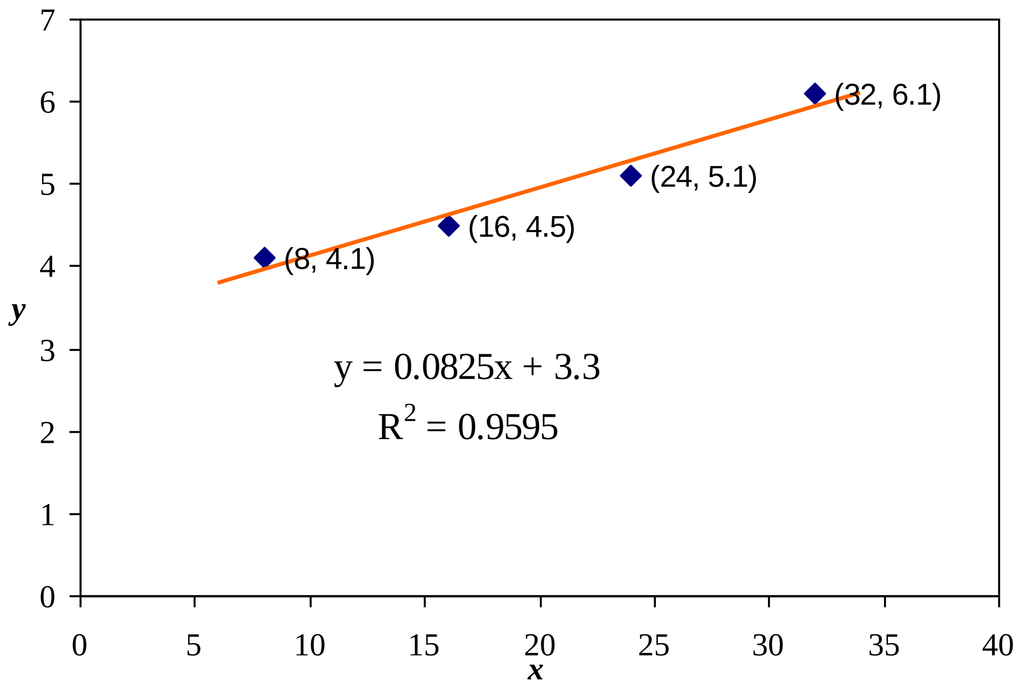
$$\boxed{\hat{a} = \frac{\left(\sum_n x_i^2\right)\left(\sum_n y_i\right) - \left(\sum_n x_i\right)\left(\sum_n x_i y_i\right)}{n\sum_n x_i^2 - \left(\sum_n x_i\right)^2}}$$

**Ex. 42:** On a farm, an agronomist finds the amount of water applied (inches) and the yield of rice are given as

| | | | | | |
|---|---|---|---|---|---|
| Water | $x$ | 8 | 16 | 24 | 32 |
| Yield | $y$ | 4.1 | 4.5 | 5.1 | 6.1 |

Find an equation of the regression line of $y$ on $x$ and predict when $x$ = 12, the value of $y$.

**Answers from Excel:**




***n* = 4**

$$\sum_{i=1}^{n} y_i = 4.1 + 4.5 + 5.1 + 6.1 = 19.8$$

$$\sum_{i=1}^{n} x_i = 8 + 16 + 24 + 32 = 80$$

$$\sum_{i=1}^{n} x_i y_i = (8)(4.1) + (16)(4.5) + (24)(5.1) + (32)(6.1) = 422.4$$

$$\sum_{i=1}^{n} x_i^{\ 2} = 8^2 + 16^2 + 24^2 + 32^2 = 1920$$

$$\hat{b} = \frac{\sum_n x_i \sum_n y_i - n\sum_n x_i y_i}{\left(\sum_n x_i\right)^2 - n\sum_n x_i^{\ 2}} = \frac{(80)(19.8) - (4)(422.4)}{(80)^2 - 4(1920)} = \frac{1584 - 1689.6}{6400 - 7680} = \frac{-105.6}{-1280} = 0.0825$$

$$\hat{a} = \frac{\left(\sum_n x_i^{\ 2}\right)\left(\sum_n y_i\right) - \left(\sum_n x_i\right)\left(\sum_n x_i y_i\right)}{n\sum_n x_i^{\ 2} - \left(\sum_n x_i\right)^2} = \frac{(1920)(19.8) - (80)(422.4)}{4(1920) - (80)^2}$$

$$\hat{a} = \frac{38016 - 33792}{7680 - 6400} = \frac{4224}{1280} = 3.3$$

Hence, the regression line is

$$\hat{y} = \hat{a} + \hat{b}x \quad \Rightarrow$$

$$y = 3.3 + 0.0825x$$

When $x = 12$,

$$y = 3.3 + 0.0825(12) = 3.3 + 0.99 = 4.29$$

**Ex. 43:** A manufacturer wants to approximate the cost function of a product. The value of the cost function has been determined for certain levels of production as listed in table below.

| | | | | |
|---|---|---|---|---|
| Number of Units $x$ *(hundreds)* | 2 | 5 | 6 | 9 |
| Cost $y$ *(thousand ฿)* | 4 | 6 | 7 | 8 |

Estimate the cost when 1,000 units is produced.

[Ans: $y = 0.58x + 3.06$, 8.86]

**Ex. 44:** The table below lists the midterm and final examination scores for 10 students in MA217 course.

| Midterm | Final | Midterm | Final |
|---|---|---|---|
| 49 | 61 | 78 | 77 |
| 53 | 47 | 83 | 81 |
| 67 | 72 | 85 | 79 |
| 71 | 76 | 91 | 93 |
| 74 | 68 | 99 | 99 |

(A) Find the line of regression of the above data.
(B) Predict the final examination score for a student who scored 95 on the midterm examination.

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Dear students,

If you find any mistake in the handout or if you have suggestion, I will be very pleased to get some feedback. Just tell me or e-mail me at anchalm@mtec.or.th.

Regards,
Anchalee