Solution: Exercise 6

Applications of Derivative: Optimization

1. A box with a square base is to be constructed. Suppose that only 10 m^2 of material is available to use in construction of the box. Assume that all the material is used in the construction process. Determine the maximum volume that the box can have.

   Solution: Let w be the length of the square base and h be the height of the box. Then the volume is given by
   $$V = w^2 h$$
   and since we must use total material 10 m^2, we have to set the total surface area of the box(2 bases and 4 sides):
   $$10 = 2w^2 + 4wh \quad \Longrightarrow \quad h = \frac{10-2w^2}{4w} = \frac{5-w^2}{2w}$$
   and $V(w) = w^2\left(\frac{5-w^2}{2w}\right)$. That is,
   $$V(w) = \frac{5w-w^3}{2}$$
   Since
   $$V'(w) = \frac{5-3w^2}{2}$$
   and $V'(w) = 0$ when $w = \pm\sqrt{\frac{5}{3}}$, then the only critical number is $w = sqrt\frac{5}{3}$ (because $w > 0$). Note that
   $$V''(w) = -3w \Longrightarrow \quad V''(\sqrt{5/3}) = -3\sqrt{5/3} < 0$$
   which implies (by the Second Derivative Test) that a relative maximum occurs at $w = \sqrt{\frac{5}{3}}$. Since $w = \sqrt{\frac{5}{3}}$ is only one critical number and it gives a relative maximum, it also gives the absolute maximum as well. Therefore, the maximum volume is $V(\sqrt{5/3}) = \frac{1}{2}(5(\sqrt{5/3}) - (\sqrt{5/3})^3) \approx 2.157m^3$

2. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50ft^3, determine the dimensions that will minimize the cost to build the box. What is the corresponding minimum cost of this box?

   Solution: Let l, w, h be the length, width, and height of the box, respectively. Total cost C = 10(Area of bases: Top and Bottom) + 6(Area of 4 Sides):
   $$C = 10(2lw) + 6(2hw + 2hl).$$
   Since we must have $l = 3w$ and $hlw = 50$ (the box must have a volume of 50ft^3) then $h = 50/3w^2$ and
   $$C(w) = 20(3w^2) + 12(50/3w^2)w + 12(50/3w^2)(3w) \Longrightarrow \quad C(w) = 60w^2 + \frac{800}{w}$$

To find critical numbers, set

$$C'(w) = 120w - \frac{800}{w^2} = 0 \quad \Longrightarrow \quad w = \left(\frac{20}{3}\right)^{1/3}$$

and the only critical number is $w = \left(\frac{20}{3}\right)^{1/3}$. Since

$$C''(w) = 120 + \frac{1600}{w^3} \quad \Longrightarrow \quad C''(\left(\frac{20}{3}\right)^{1/3}) > 0$$

then a relative minimum occurs at $w = \left(\frac{20}{3}\right)^{1/3}$ (by the Second Derivative Test). Since $w = \left(\frac{20}{3}\right)^{1/3}$ is only one critical number and it gives a relative minimum , it also gives the absolute minimum. That is, the dimension that minimizes the cost of this box is $w = \left(\frac{20}{3}\right)^{1/3} \approx 1.8821$ ft, $l = 3\left(\frac{20}{3}\right)^{1/3} \approx 5.6463$ ft, and $h = \frac{50}{3}\left(\frac{20}{3}\right)^{-2/3} \approx 4.7050$ ft.
The minimum cost is $C(\left(\frac{20}{3}\right)^{1/3}) \approx \$637.60$.

3. We want to print a poster that has a total area of 200 in^2 with 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. Determine the dimensions of the poster that will give the largest printed area.

   Solution: Let x and y be the width and height of the poster. Then we must have

   $$yx = 200.$$

   Let A be the printed area. Then $A = (y-2-1.5)(x-1-1) = (y-3.5)(x-2)$. Since $y = 200/x$

   $$A(x) = (\frac{200}{x} - 3.5)(x-2).$$

   Since

   $$A'(x) = -3.5 + \frac{400}{x^2} = \frac{400 - 3.5x^2}{x^2}$$

   and $x > 0$, the critical number is x with $400 - 3.5x^2 = 0 \Longrightarrow x = \sqrt{\frac{400}{3.5}}$.

   $$A''(x) = -\frac{800}{x^3} \quad \Longrightarrow \quad A''(\sqrt{(400/3.5)}) < 0$$

   which implies (by the Second Derivative Test) that a relative maximum occurs at $x = \sqrt{\frac{400}{3.5}}$. Since $x = \sqrt{\frac{400}{3.5}}$ is only one critical number and it gives a relative maximum, it also gives the absolute maximum as well. The dimension that maximizes the area is $x = \sqrt{\frac{400}{3.5}} \approx 10.9604$ inches and $y = 200\sqrt{\frac{3.5}{400}} \approx 18.7084$ inches.

4. At midnight, ship A is 50 km north of ship B. Ship A is sailing south at 20 km/h. Ship B is sailing west at 10 km/h. At what time will the distance between the ships be a minimum?

   Solution:

Let $x(t)$ and $y(t)$ be the distances of Ship B and Ship A from the initial starting position of Ship B. Then $x(t) = 10t$ and $y(t) = 50 - 20t$, for time $t \geq 0$ and the distance between Ship A and Ship B is

$$d(t) = \sqrt{(10t)^2 + (50 - 20t)^2} \Longrightarrow D(t) := d(t)^2 = (10t)^2 + (50 - 20t)^2.$$

We want to find t that minimize $D(t)$ (and hence $d(t)$).

$$D'(t) = 200t - 2(50 - 20t)(-20) = 200t - 2000 + 800t = 1000t - 2000.$$

We first find the critical number

$$D'(t) = 0 \quad \Longrightarrow \quad 1000t - 2000 = 0 \quad \Longrightarrow \quad t = 2.$$

That is there is only one critical number at $t = 2$. Note that

$$D''(t) = 1000 \quad \Longrightarrow \quad D''(2) = 1000 > 0$$

and therefore, by the Second Derivative Test, a relative minimum occurs at $t = 2$. Since $t = 2$ is the only critical number and it gives a relative minimum, then it also gives the absolute minimum. That is, the minimum distance $d(t)$ occurs when $t = 2$ or at 2 AM.

5. Determine the area of the largest rectangle that can be inscribed in a circle of radius 4 cm.

Solution: Let w be the width of the rectangle and ℓ be the length of the rectangle. Sice this rectangle is inscribed in a circle of radius 4 cm (diameter 8 cm), we must have

$$w^2 + \ell^2 = 8^2 \quad \Longrightarrow \quad \ell = \sqrt{64 - w^2}.$$

We want to maximize the area $A = w\ell = w\sqrt{64 - w^2}$ or

$$A(w) = w\sqrt{64 - w^2}.$$

Note $A'(w) = w\frac{1}{2}(64 - w^2)^{-1/2}(-2w) + (64 - w^2)^2 = -\frac{w}{\sqrt{64 - w^2}} + \sqrt{64 - w^2}$ or

$$A'(w) = -\frac{w^2}{\sqrt{64 - w^2}} + \sqrt{64 - w^2} \quad \Longrightarrow \quad A'(w) = \frac{64 - 2w^2}{\sqrt{64 - w^2}}.$$

We first find critical number of $A(w)$, $w > 0$:

$$A'(w) = 0 \Longrightarrow 64 - 2w^2 = 0 \Longrightarrow w^2 = 64 - w^2 \Longrightarrow w = \sqrt{32} = 4\sqrt{2},$$

so $w = 4\sqrt{2}$ is the only critical number. Note that

$$A''(w) = \frac{-4w\sqrt{64 - w^2} - 2w^2\frac{1}{2}(64 - w^2)^{-1/2}(-2w)}{(\sqrt{64 - w^2})^2} = \frac{-4w\sqrt{64 - w^2} + 2w^3(64 - w^2)^{-1/2}}{64 - w^2}.$$

$$A''(w) = 2\frac{-2w(64 - w^2) + w^3}{(64 - w^2)^{3/2}} = 2w\frac{-128 + 3w^2}{(64 - w^2)^{3/2}}.$$

and $A''(\sqrt{32}) < 0$ (because $-128 + 3(\sqrt{32})^2 = 128 - 96 < 0$) and a relative maximum occurs at $w = \sqrt{32}$, by the Second Derivative Test. Since $w = \sqrt{32} = 4\sqrt{2}$ is only one critical number and it gives a relative maximum, it also gives the absolute maximum as well. That is, $w = \sqrt{32} \Longrightarrow \ell = \sqrt{64 - w^2} = \sqrt{32}$ maximize the area and the area of the largest rectangle that can be inscribed in a circle of radius 4 cm is $w\ell = 32$ cm^2.

6. Find the point on the graph of $y^2 = 2x$ closest to $(2,0)$.

   Solution: Let $d = \sqrt{(x-2)^2 + (y-0)^2}$ be the distance between the point (x,y) and the point $(2,0)$. For (x,y) on the graph $y^2 = 2x$, we have

   $$d = \sqrt{(x-2)^2 + y^2} = \sqrt{(x-2)^2 + 2x} \quad \Longrightarrow \quad d(x) = \sqrt{(x-2)^2 + 2x}$$

   Let $D(x) = d(x)^2$ or $D(x) = (x-2)^2 + 2x$. Note that x minimizes $d(x)$ if and only if it minimizes $D(x)$. So we will use $D(x)$ to find the value of x that minimize $d(x)$. Note

   $$D'(x) = 2(x-2) + 2 = 2x - 2$$

   First, we find the critical point:

   $$D'(x) = 0 \Longrightarrow 2x - 2 = 0 \Longrightarrow x = 1.$$

   That is there is only one critical number $x = 1$. Since $D''(x) = 2$, then $D''(1) = 2 > 0$ and the critical number $x = 1$ gives a relative minimum (by the Second Derivative Test). Since $x = 1$ is only one critical number and it gives a relative minimum, it also gives the absolute minimum as well. When $x = 1$, $y^2 = 2 \Longrightarrow y = \pm\sqrt{2}$ That is, the points on the graph of $y^2 = 2x$ closest to $(2,0)$ occur at $(1, \sqrt{2})$ and $(1, -\sqrt{2})$.