Two-Variable Regression Model: The Problem of Estimation

The Two-Variable PRF:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

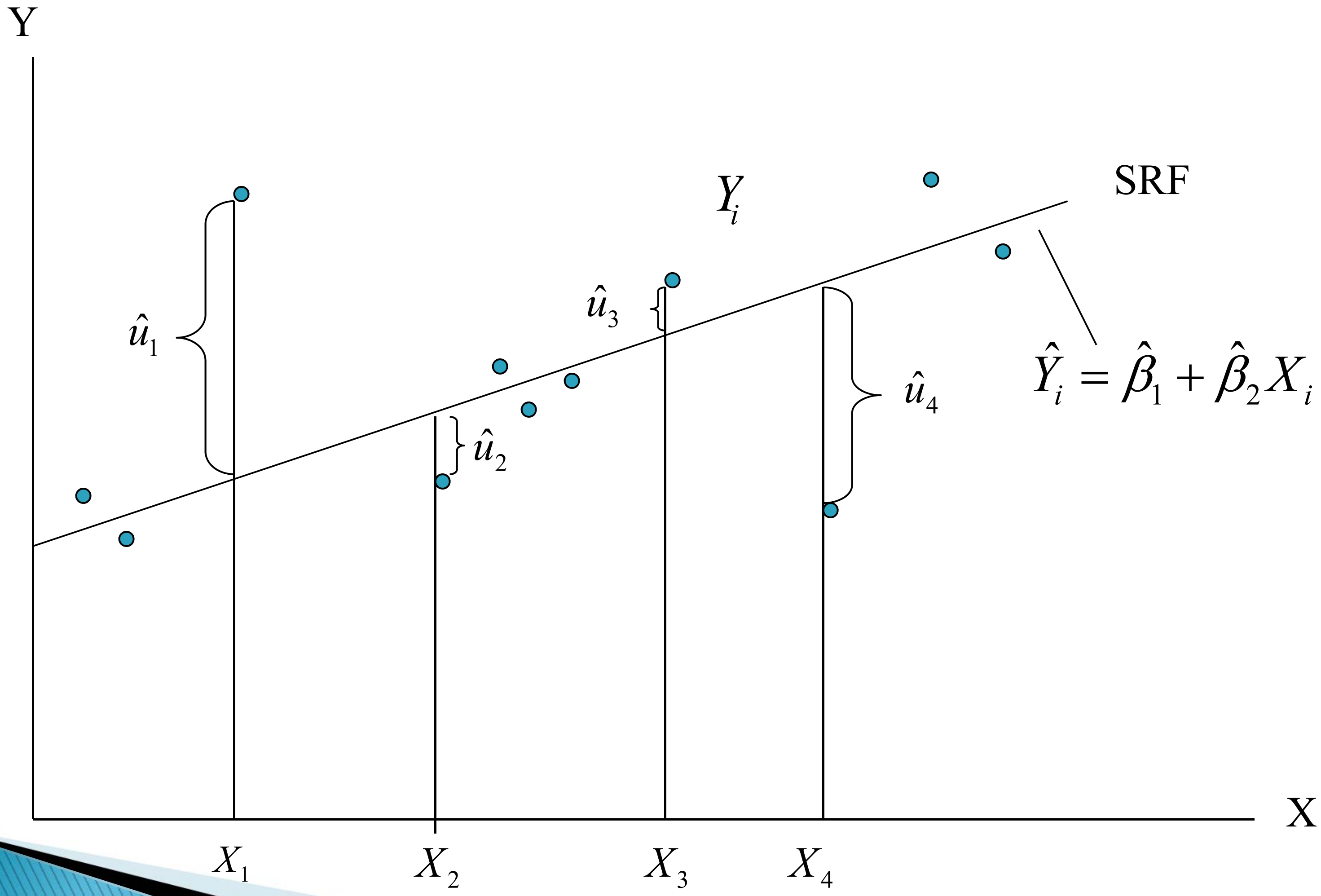
The Two-Variable SRF:

$$\begin{aligned} Y_i &= \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \\ &= \hat{Y}_i + \hat{u}_i \end{aligned}$$

$\hat{Y}_i$ is the estimated (conditional mean) value of Y_i

$$
\begin{aligned}
\hat{u}_i &= Y_i - \hat{Y}_i \\
&= Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i
\end{aligned}
$$

Ordinary Least Squares (OLS)



Ordinary Least Squares (OLS)

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{Y}_i)^2$$
$$= \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2$$

The principle or the method of least squares chooses $\hat{\beta}_1 and \hat{\beta}_2$ in such a manner that, for a given sample or set of data, $\sum \hat{u}_i^2$ is as small as possible

$$\sum \hat{u}_i^{\,2} = \mathrm{f}(\hat{\beta}_1, \hat{\beta}_2)$$

$$\frac{\partial\left(\sum \hat{u}_i^2\right)}{\partial \hat{\beta}_1} = -2\sum\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) = -2\sum \hat{u}_i$$

$$\frac{\partial\left(\sum \hat{u}_i^2\right)}{\partial \hat{\beta}_2} = -2\sum\left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i\right) X_i = -2\sum \hat{u}_i X_i$$

Setting these equation to zero

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad \text{PRF}$$

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \quad \text{SRF}$$

$$= \hat{Y}_i + \hat{u}_i$$

$$\hat{u}_i = Y_i - \hat{Y}_i$$

$$= Y_i - \hat{\beta}_1 + \hat{\beta}_2 X_i$$

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{Y}_i)^2$$

$$= \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2$$

$$\frac{\partial(\sum \hat{u}_i)}{\partial \hat{\beta}_1} = -2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i) = 0$$

$$\sum Y_i - \sum \hat{\beta}_1 - \sum \hat{\beta}_2 X_i = 0$$

$$\sum Y_i - n\hat{\beta}_1 - \hat{\beta}_2 \sum X_i = 0$$

$$\sum Y_i - \hat{\beta}_2 \sum X_i = n\hat{\beta}_1$$

$$\frac{\sum Y_i}{n} - \hat{\beta}_2 \frac{\sum X_i}{n} = \hat{\beta}_1$$

$$\bar{Y}_i - \hat{\beta}_2 \bar{X} = \hat{\beta}_1$$

$$\bar{X} = \frac{1}{n} \sum X_i \Rightarrow \sum X_i = n\bar{X}$$

$$\bar{Y} = \frac{1}{n} \sum Y \Rightarrow \sum Y = n\bar{Y}$$

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$$

$$\frac{\partial(\Sigma \hat{u}_i)^2}{\partial \hat{\beta}_2} = -2\Sigma(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)(X_i)$$

$$= \Sigma Y_i X_i - \hat{\beta}_1 \Sigma X_i - \hat{\beta}_2 \Sigma X_i^2$$

$$= \Sigma Y_i X_i - (\bar{Y} - \hat{\beta}_2 \bar{X}) \Sigma X_i - \hat{\beta}_2 \Sigma X_i^2$$

$$= \Sigma Y_i X_i - \bar{Y} \Sigma X_i + \hat{\beta}_2 \bar{X} \Sigma X_i - \hat{\beta}_2 \Sigma X_i^2$$

$$= \Sigma Y_i X_i - \frac{\Sigma Y_i}{n} \Sigma X_i + \hat{\beta}_2 \frac{\Sigma X_i}{n} \Sigma X_i - \hat{\beta}_2 \Sigma X_i^2$$

$$= n\Sigma Y_i X_i - \Sigma Y_i \Sigma X_i + \hat{\beta}_2 \Sigma X_i^2 - \hat{\beta}_2 n \Sigma X_i^2$$

$$\hat{\beta}_2 n \Sigma X_i^2 - \hat{\beta}_2 \Sigma X_i^2 = n\Sigma Y_i X_i - \Sigma Y_i \Sigma X_i$$

$$\hat{\beta}_2 = \frac{n\Sigma Y_i X_i - \Sigma Y_i \Sigma X_i}{n\Sigma X_i^2 - \Sigma X_i^2} \checkmark$$

$$\sum Y_i = n\hat{\beta}_1 + \hat{\beta}_2 \sum X_i$$

$$\sum Y_i X_i = \hat{\beta}_1 \sum X_i + \hat{\beta}_2 \sum X_i^2$$

Where n is the sample size. These simultaneous equations are known as the **Normal Equation**

$$\hat{\beta}_2 = \frac{n\sum X_i Y_i - \sum X_i \sum Y_i}{n\sum X_i^2 - (\sum X_i)^2}$$

$$= \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$

$$= \frac{\sum x_i y_i}{\sum x_i^2}$$

$$\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$= \frac{\sum x_i Y_i}{\sum X_i^2 - n\bar{X}^2}$$

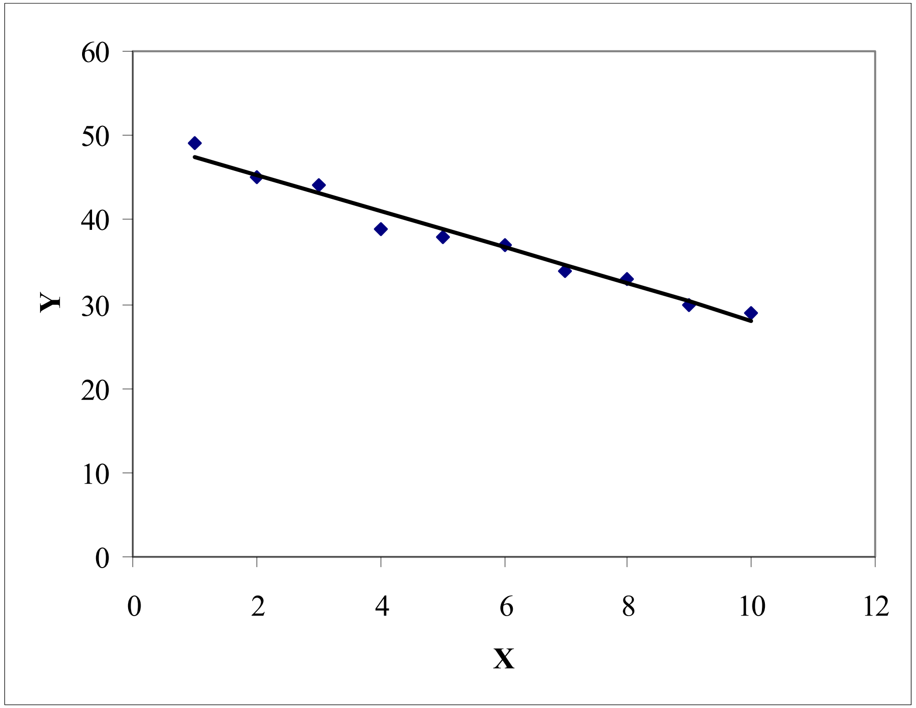
$$= \frac{\sum X_i y_i}{\sum X_i^2 - n\bar{X}^2}$$

$$\hat{\beta}_1 = \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_i Y_i}{n \sum X_i^2 - (\sum X_i)^2}$$

$$= \bar{Y} - \hat{\beta}_2 \bar{X}$$

Example ☺

Y	X
49	1
45	2
44	3
39	4
38	5
37	6
34	7
33	8
30	9
29	10



$$\hat{\beta}_2 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{-178}{82.5} \approx -2.1576$$

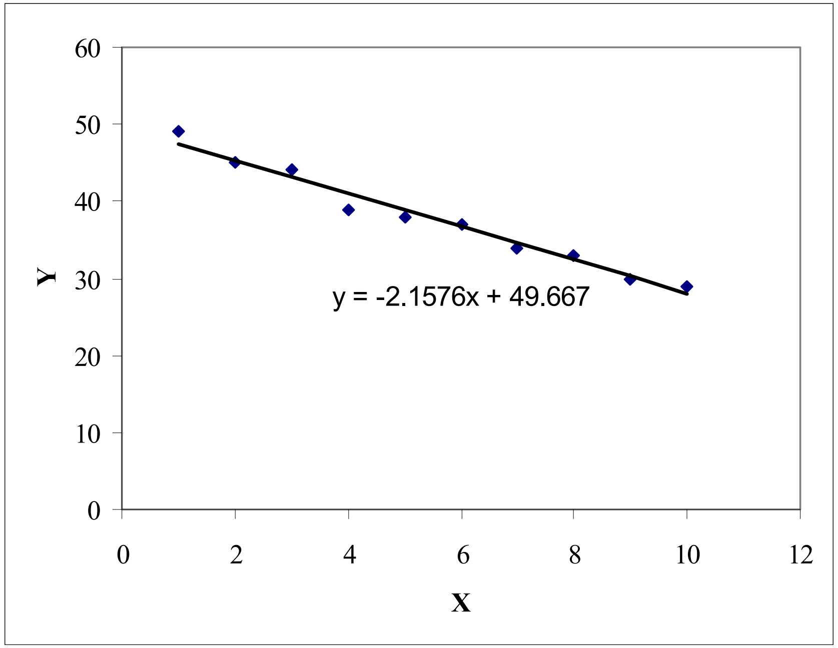
$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 37.8 - (-2.1576)(5.5) = 49.667$$

$$\bar{X} = 5.5$$

$$\bar{Y} = 37.8$$

Y	X	$(X_i - \bar{X})$	$(X_i - \bar{X})^2$	$Y_i - \bar{Y}$	$(X_i - \bar{X})(Y_i - \bar{Y})$
49	1	-4.5	20.25	11.2	-50.4
45	2	-3.5	12.25	7.2	-25.2
44	3	-2.5	6.25	6.2	-15.5
39	4	-1.5	2.25	1.2	-1.8
38	5	-0.5	0.25	0.2	-0.1
37	6	0.5	0.25	-0.8	-0.4
34	7	1.5	2.25	-3.8	-5.7
33	8	2.5	6.25	-4.8	-12
30	9	3.5	12.25	-7.8	-27.3
29	10	4.5	20.25	-8.8	-39.6
			82.5		-178

$$\hat{Y}_i = 49.667 - 2.1576 X_i$$



Practice I

A Random Sample from the Population of Table 2.1

Weekly consumption expenditure $ (Y)	Weekly income $(X)
70	80
65	100
90	120
95	140
110	160
115	180
120	200
140	220
155	240
150	260

$$\hat{Y} = 0.5091X_i + 24.455$$

Practice II

Another Random Sample from the Population of Table 2.1

Weekly consumption expenditure $ (Y)	Weekly income $(X)
Y	X
55	80
88	100
90	120
80	140
118	160
120	180
145	200
135	220
145	240
175	260

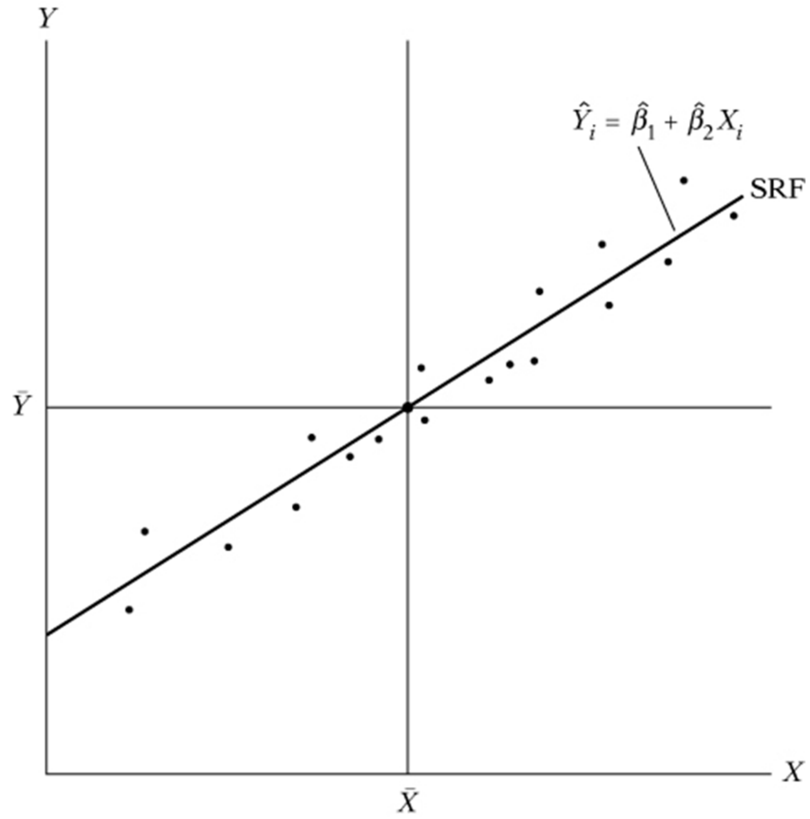
$$\hat{Y} = 0.5761X_i + 17.17$$

- The estimators obtained are known as the least-squares estimators
- The statistical properties of OLS estimators – properties that hold only under certain assumptions about the way the data were generated
  - The OLS estimators are expressed solely in terms of the observable (i.e. sample) quantities (i.e., X and Y)
  - They are point estimators
  - Once the OLS estimates are obtained from the sample data, the sample regression line can be easily obtained.

The regression line has the following properties:

①

- Passes through the sample means of Y and X



$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$

(2)

- The mean value of the estimated $Y = \hat{Y}_i$ is equal to the mean value of the actual Y for

$$\frac{\sum \hat{Y}_i}{n} = \frac{\sum \bar{Y}}{n}$$

$$\bar{\hat{Y}} = \bar{Y}$$

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$$

$$= (\bar{Y} - \hat{\beta}_2 \bar{X}) + \hat{\beta}_2 X_i$$

$$= \bar{Y} + \hat{\beta}_2 (X_i - \bar{X})$$

Summing both sides

$$\sum \hat{Y}_i = \sum \bar{Y} + \hat{\beta}_2 \sum (X_i - \bar{X})$$

$\sum (X_i - \bar{X}) \to 0$

③

- The mean value of the residuals $\hat{u}_i$ is zero

④

- The residuals $\hat{u}_i$ are uncorrelated with the predicted Y_i

⑤

- The residuals $\hat{u}_i$ are uncorrelated with X_i

③

$$-2\sum(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i) = 0$$

$$\hat{u}_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i$$

$$-2\sum \hat{u}_i = 0$$

$$\bar{\hat{u}} = 0$$ //

④ $\sum \hat{y}\hat{u}_i = \hat{\beta}_2 \sum x\hat{u}_i$

$= \hat{\beta}_2 \sum x_i (y_i - \hat{\beta}_2 x_i)$

$= \hat{\beta}_2 \sum x_i y_i - \hat{\beta}_2^2 \sum x_i^2$

where use is made of the fact that $\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2}$

$= \hat{\beta}_2^2 \sum x_i^2 - \hat{\beta}_2^2 \sum x_i^2$

$= 0$

(5) The residual $\hat{u}_i$ are uncorrelated with X_i; that is $\sum \hat{u}_i X_i = 0$

$$\frac{\partial(\sum \hat{u}_i^2)}{\partial \hat{\beta}_2} = -2\sum(Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)X_i$$

$$= -2\sum \hat{u}_i X_i = 0$$

Classical Linear regression model (CLRM)

The assumptions underlying the method of least squares:

1. Linear regression model
2. Fixed X values or X values independent of the error u_i term
3. Zero mean value of disturbance u_i
4. Homoscedasticity or Constant Variance of
5. No autocorrelation between the disturbances
6. The number of observations n must be greater than the number of parameters to be estimated
7. The nature of X variables

Linear regression model

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

- Linear in the parameters
- May or may not be linear in the variables

" Fixed X values or X values are nonstochastic"

X values independent of the error term

$$Cov(X_i, u_i) = 0$$

We assume that X variable(s) is nonstochastic for the following reasons:

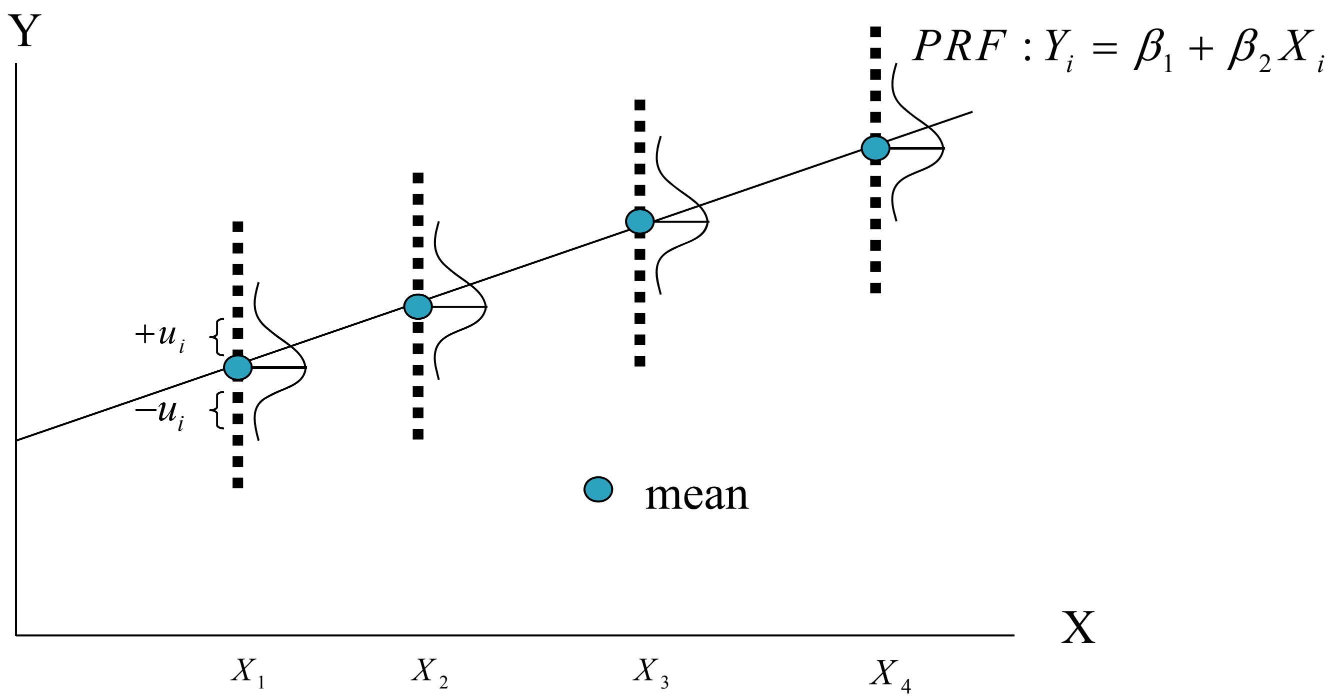
① To simplify the analysis

② in experimental

Zero mean value of disturbance term

$$E(u_i \mid X_i) = 0$$ ← implies that $E(Y_i|X_i) = \beta_1 + \beta_2 X_i$

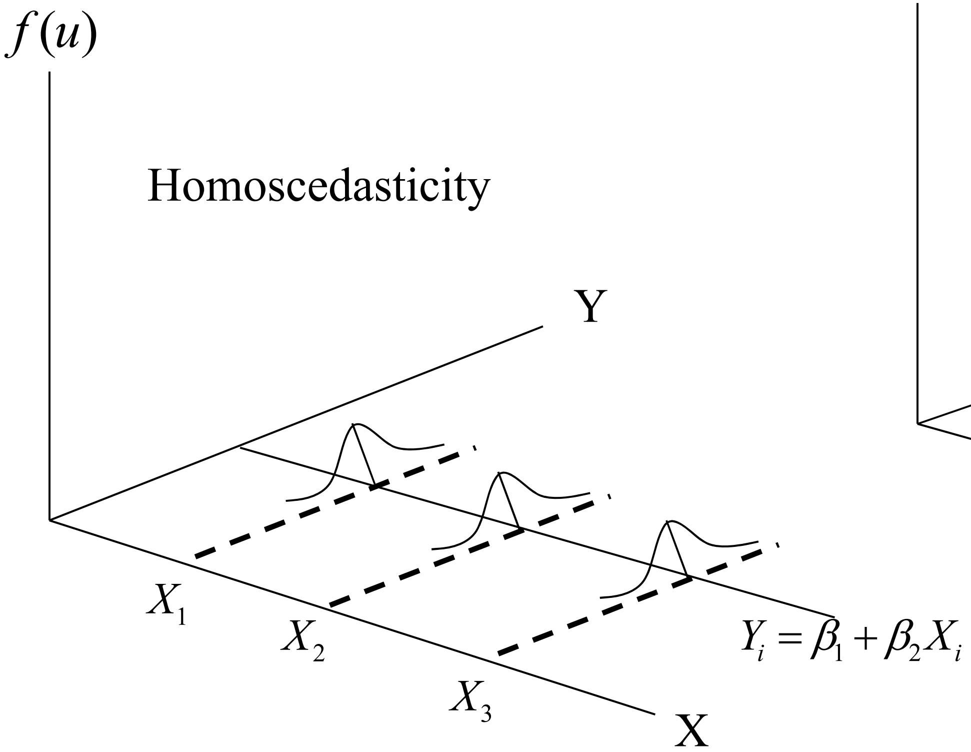
$$E(u_i) = 0$$



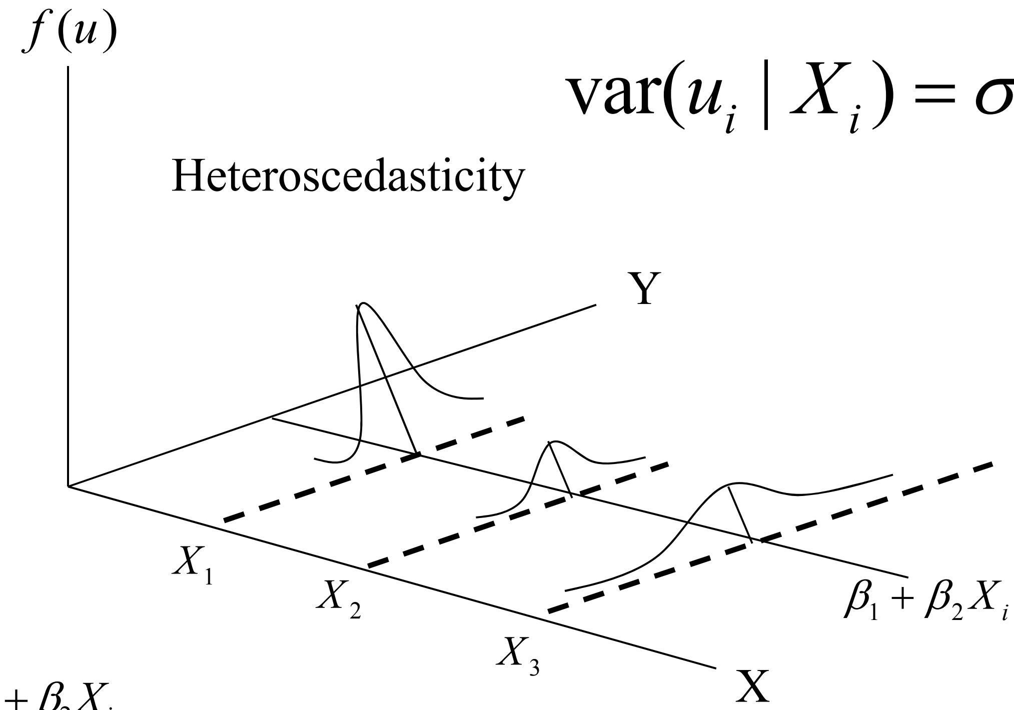
Homoscedasticity

$$\begin{aligned}\operatorname{var}(u_i) &= E[u_i - E(u_i \mid X_i)]^2 \\ &= E(u_i^2 \mid X_i), \textit{because of assumption } 3 \\ &= E(u_i^2), \textit{if } X_i \textit{ are nonstochastic} \\ &= \sigma^2\end{aligned}$$

Homoscedasticity vs. Heteroscedasticity



$$\mathrm{var}(u_i \mid X_i) = \sigma^2$$



$$\mathrm{var}(u_i \mid X_i) = \sigma_i^2$$

Homoscedasticity v.s. Heteroscedasticity

Homoscedasticity

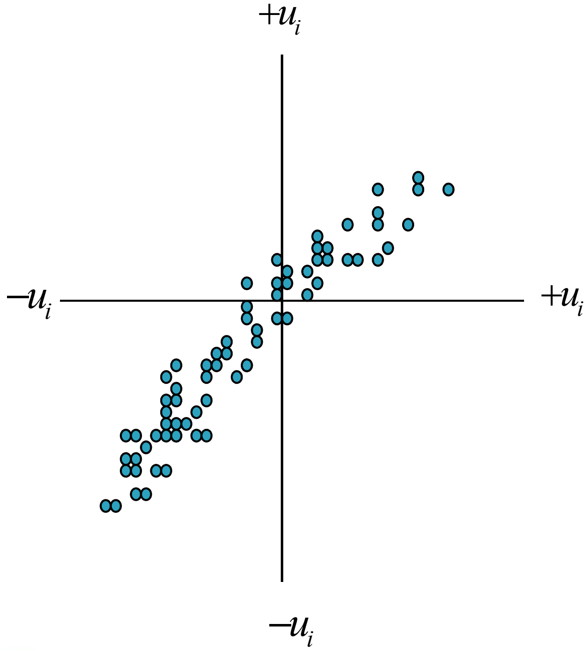
- Equal variance
- The variation around the regression line is the same across the X values

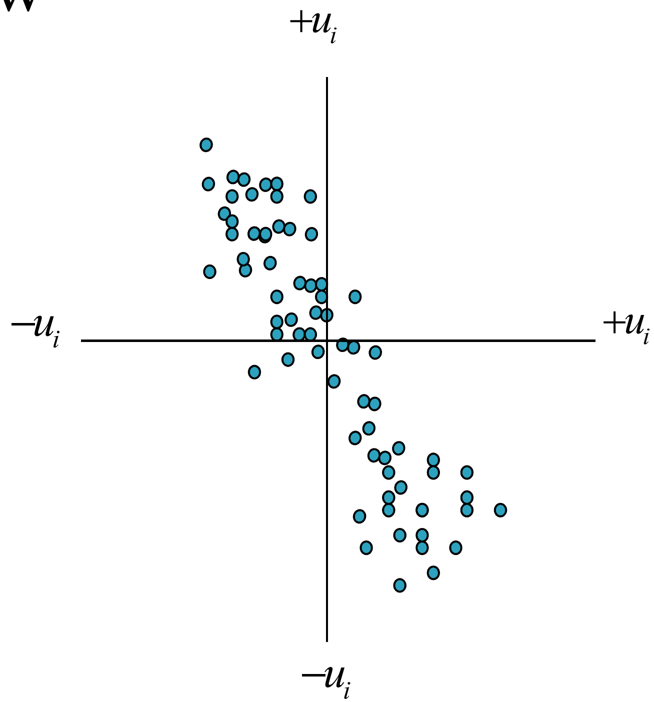
Heteroscedasticity

- Unequal variance

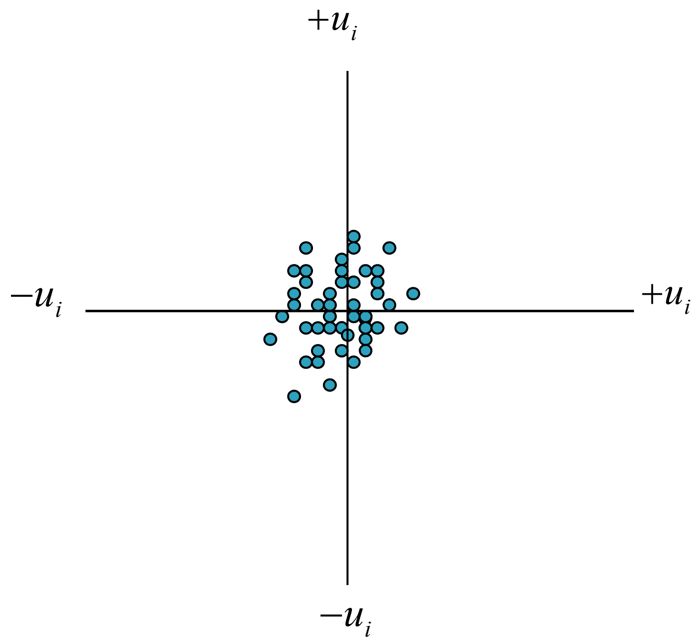
No autocorrelation between the disturbances

Given X_i , the deviations of any two Y values from their mean value do not exhibit patterns such as those shown in figures below





No autocorrelation between the disturbances



No autocorrelation between the disturbances

Given any two X values, X_i and $X_j (i \neq j)$, the correlation between any two u_i and $u_j (i \neq j)$ is zero.

$$\mathrm{cov}(u_i, u_j \mid X_i, X_j) = 0$$

$$\mathrm{cov}(u_i, u_j) = 0, \text{ if } X \text{ is nonstochastic}$$

Where i and j are two different observation and where cov means covariance

The number of observations n must be greater than the number of parameters to be estimated

The number of observations must be greater than the number of explanatory variables

The nature of X variables

- The X values in a given sample must not all be the same

- No outliers in the X values

Standard Errors of Least-Squares Estimates

$$\operatorname{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2}$$

$$se(\hat{\beta}_2) = \frac{\sigma}{\sqrt{\sum x_i^2}}$$

$$\operatorname{var}(\hat{\beta}_1) = \frac{\sum X_i^2}{n\sum x_i^2}\sigma^2$$

$$se(\hat{\beta}_1) = \sqrt{\frac{\sum X_i^2}{n\sum x_i^2}}\sigma$$

$$Y_i = \underbrace{\hat{\beta}_1 + \hat{\beta}_2 X_i}_{} + \hat{u}_i$$

$$Y_i = \hat{Y}_i + \hat{u}_i$$ ← Error term

True Y (↑ Y_i)

Estimated Y (↑ $\hat{Y}_i$) ← Deterministic term

$\sum (Y_i - \hat{Y}_i)^2$ → $\sum \hat{u}_i^2$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2}$$

$\hat{\sigma}^2$ is the OLS estimator of the true but unknown σ^2

The expression n-2 is known as the number of degrees of freedom

$\sum \hat{u}_i^2$ is the sum of the residuals squared or residual sum of squares (RSS)

$$COV(\hat{\beta}_1, \hat{\beta}_2) = -\bar{X}\,\mathrm{var}(\hat{\beta}_2)$$

$$= -\bar{X}\left(\frac{\sigma^2}{\sum x_i^2}\right)$$

$$= \frac{-\bar{X}\sigma^2}{\sum (X_i - \bar{X})^2}$$

Example ☺

Y	X
49	1
45	2
44	3
39	4
38	5
37	6
34	7
33	8
30	9
29	10

$$\hat{\beta}_1 = 49.667$$

$$\hat{\beta}_2 = -2.1576$$

$$\bar{X} = 5.5$$

Y	X	$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$	$\hat{u}_i$	$\hat{u}_i^2$	$(X_i - \bar{X})^2$	X_i^2
49	1	47.5094	1.4906	2.2219	20.25	1
45	2	45.3518	-0.352	0.1238	12.25	4
44	3	43.1942	0.8058	0.6493	6.25	9
39	4	41.0366	-2.037	4.1477	2.25	16
38	5	38.879	-0.879	0.7726	0.25	25
37	6	36.7214	0.2786	0.0776	0.25	36
34	7	34.5638	-0.564	0.3179	2.25	49
33	8	32.4062	0.5938	0.3526	6.25	64
30	9	30.2486	-0.249	0.0618	12.25	81
29	10	28.091	0.909	0.8263	20.25	100
			Total	9.5515	82.5	385

True Y

Estimated Y

$Y - \hat{Y}_i$ Error term

$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2}$

49.667

-2.1576(1)

49-47.5094

$(\sum X_i)^2 \neq \sum X_i^2$

$$\hat{\beta}_2 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} \qquad \frac{9.5515}{10-2} = 1.1939$$

$$\operatorname{var}(\hat{\beta}_2) = \frac{\hat{\sigma}^2}{\sum x_i^2} = \frac{\hat{\sigma}^2}{\sum (X_i - \bar{X})^2} \qquad \frac{1.1939}{82.5} = 0.0145$$

$$\operatorname{var}(\hat{\beta}_1) = \frac{\sum X_i^2}{n \sum x_i^2} \hat{\sigma}^2 = \frac{\sum X_i^2}{n \sum (X_i - \bar{X})^2} \hat{\sigma}^2 \qquad \frac{(1.1939)(385)}{(10)(82.5)} = 0.5572$$

$$COV(\hat{\beta}_1, \hat{\beta}_2) = -\bar{X} \operatorname{var}(\hat{\beta}_2) \qquad -(5.5)(0.0145) = -0.07975$$

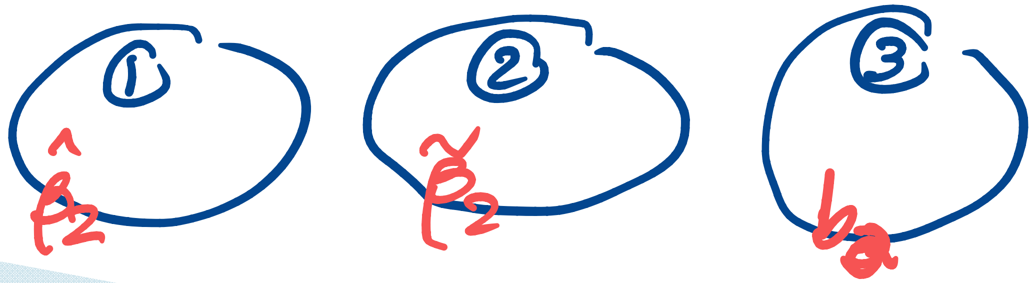
Properties of Least-Squares Estimators: The Gauss-Markov Theorem

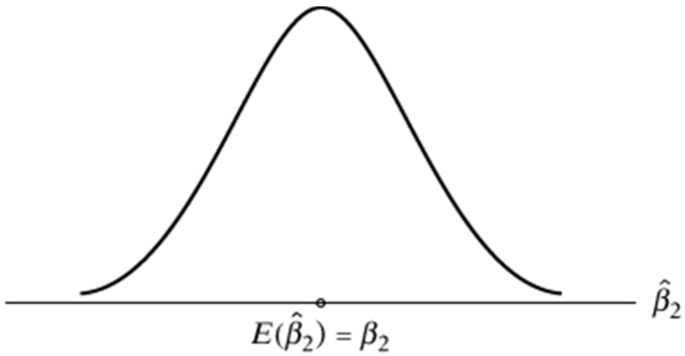
Best Linear Unbiased Estimator (BLUE) of β_2 :

- It is linear, that is, a linear function of a random variable, such as the dependent variable Y in the regression model
- It is unbiased $E(\hat{\beta}_2) = \beta_2$
- Efficient estimator – an unbiased estimator with the least variance

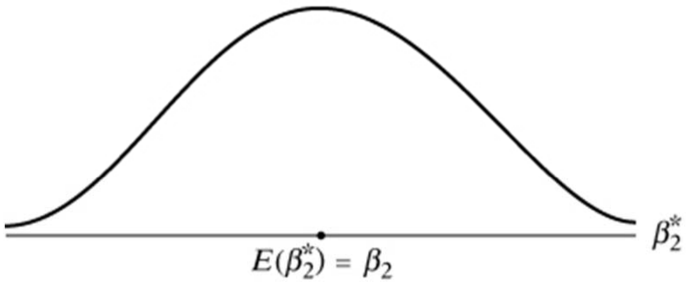
$E(\hat{\beta}_2) = \beta_2$ $\quad E(b_2) = \beta_2$

$E(\tilde{\beta}_2) = \beta_2$

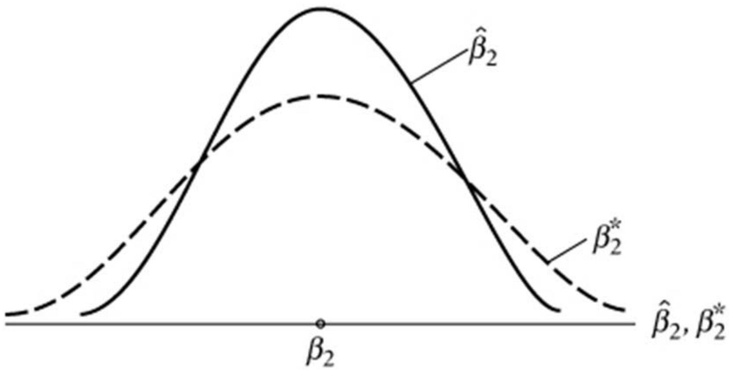




(a) Sampling distribution of β_2



(b) Sampling distribution of β_2^*



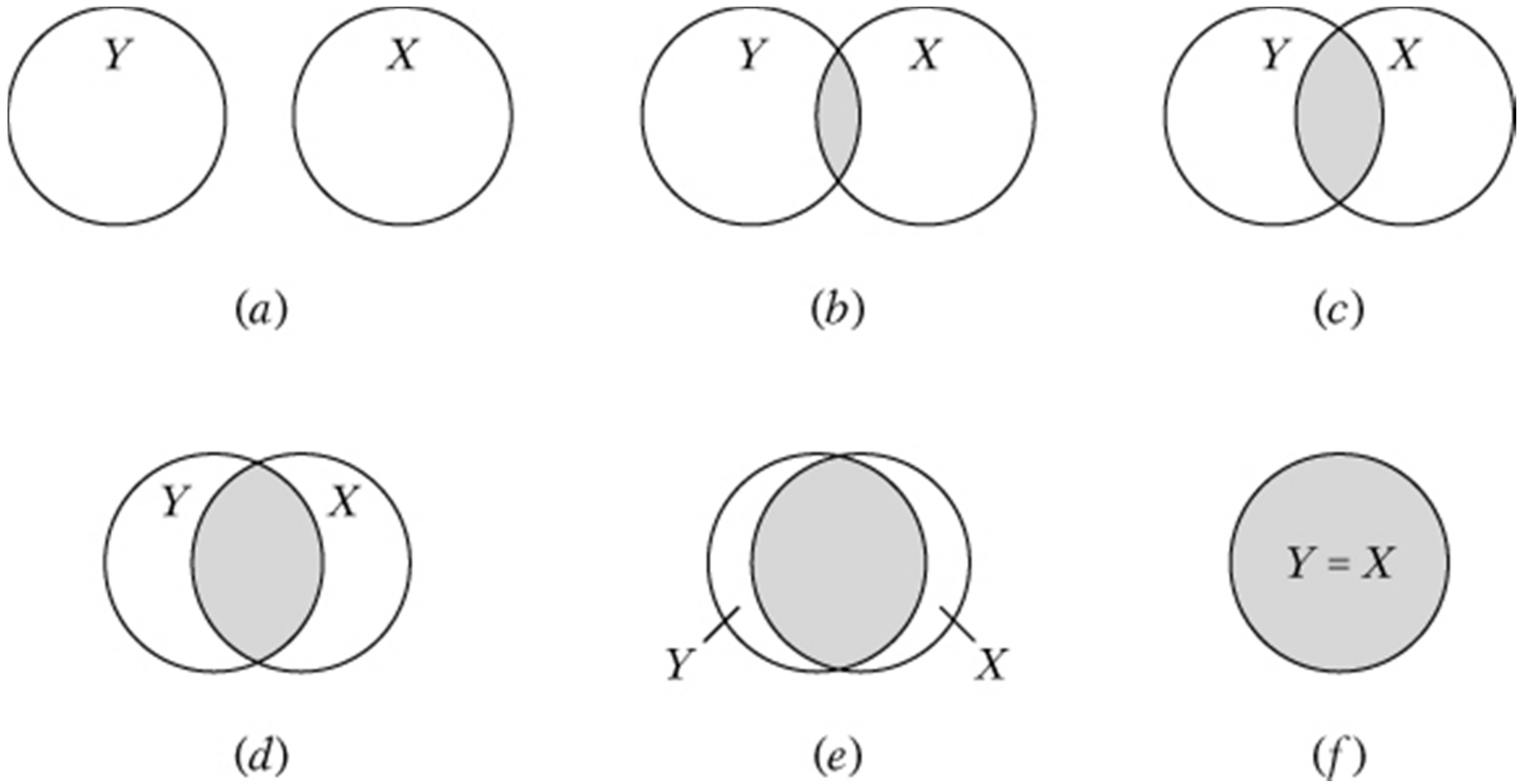
(c) Sampling distributions of β_2 and β_2^*

The Coefficient of Determination r^2

- A measure of goodness of fit

- A summary measure that tells how well the sample regression line fits the data

- Measures the proportion or percentage of the total variation in Y explained by the regression model


Y
X
(a)
Y
X
(b)
Y
X
(c)
Y
X
(d)
Y
X
(e)
Y = X
(f)

To compute this r^2

$$Y_i = \hat{Y}_i + \hat{u}_i$$ ①

$(Y_i - \bar{Y})$ → $$y_i = \hat{y}_i + \hat{u}_i$$ ②

$\hat{y}_i = (\hat{Y}_i - \bar{Y})$

$y_i^2 = (\hat{y}_i + \hat{u}_i)^2$

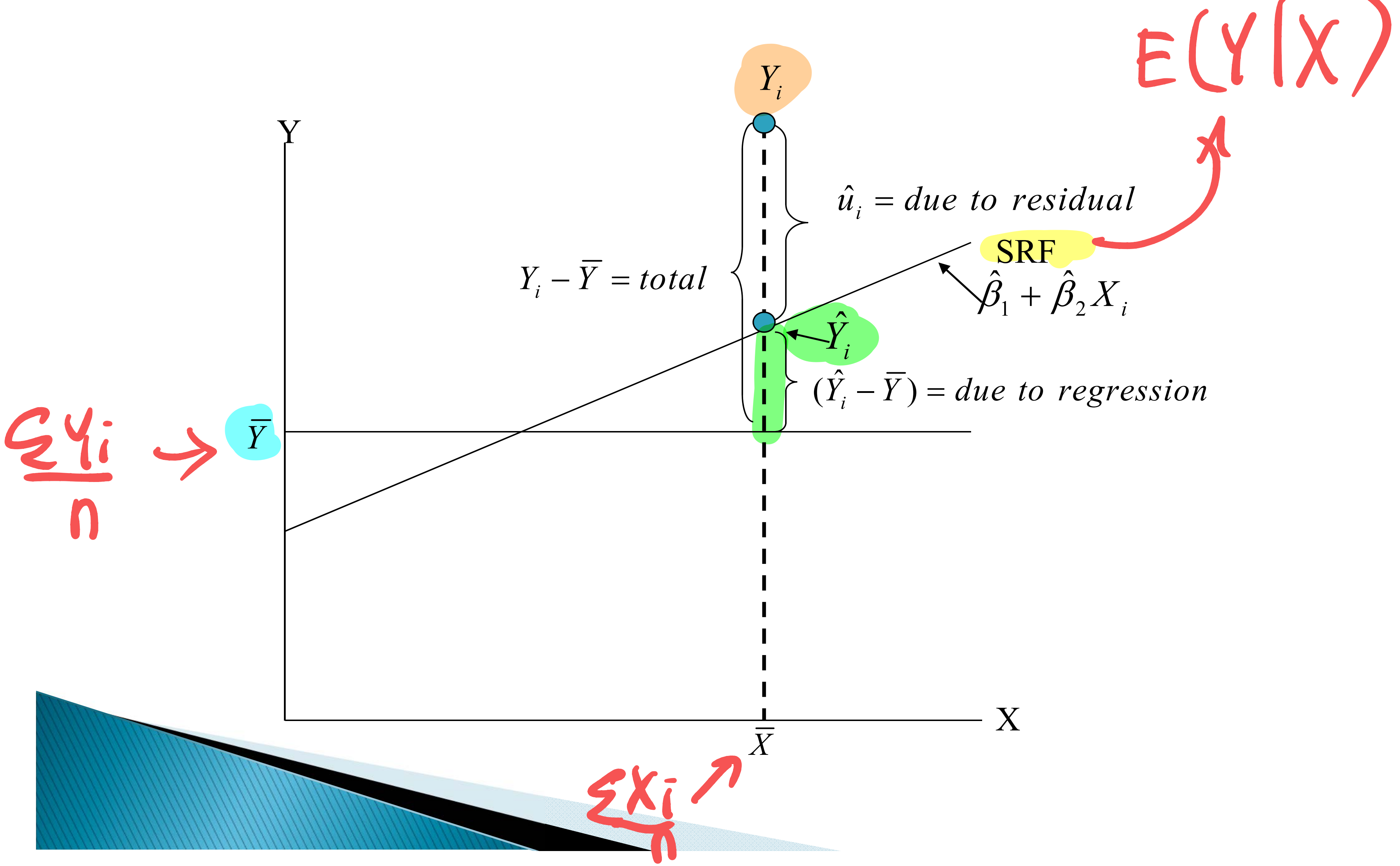
Take Σ both sides

$$\sum y_i^2 = \sum \hat{y}_i^2 + \sum \hat{u}_i^2 + 2\sum \hat{y}_i \hat{u}_i$$ ③

$$= \sum \hat{y}_i^2 + \sum \hat{u}_i^2$$

$$= \hat{\beta}_2^2 \sum x_i^2 + \sum \hat{u}_i^2$$

Sample Regression Function (SRF)



$$\sum y_i^2 = \sum (Y_i - \bar{Y})^2$$

Total variation of the actual Y values about their sample mean

(Total Sum of Squares, TSS)

$$\sum \hat{y}_i^2 = \sum (\hat{Y}_i - \hat{Y})^2 = \hat{\beta}_2^2 \sum x_i^2$$

Variation of the estimated Y values about their mean (Explained Sum of Squares, ESS)

$$\sum \hat{u}_i^2$$

Residual or unexpected variation of the Y values about the regression line (Residual Sum of Squares, RSS)

$$TSS = ESS + RSS$$

$$TSS = ESS + RSS$$ ①

$$1 = \frac{ESS}{TSS} + \frac{RSS}{TSS}$$ ②

$$= \frac{\sum (\hat{Y}_i - \bar{Y})^2}{\sum (Y_i - \bar{Y})^2} + \frac{\sum \hat{u}_i^2}{\sum (Y_i - \bar{Y})^2}$$ ③

$$r^2 = \frac{\sum (\hat{Y}_i - \bar{Y})^2}{\sum (Y_i - \bar{Y})^2} = \frac{ESS}{TSS}$$ ④

$$= 1 - \frac{RSS}{TSS} = 1 - \frac{\sum \hat{u}_i^2}{\sum (Y_i - \bar{Y})^2}$$

Example ☺

Y	X
49	1
45	2
44	3
39	4
38	5
37	6
34	7
33	8
30	9
29	10

$$\hat{\beta}_1 = 49.667$$
$$\hat{\beta}_2 = -2.1576$$
$$\bar{X} = 5.5$$
$$\bar{Y} = 38$$

$\bar{Y} = \frac{\sum Y_i}{n} = 38$

Denominator for r^2 — numerator for r^2

Y	X	$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$	$\hat{u}_i$	$(Y_i - \bar{Y})^2$	$(\hat{Y}_i - \bar{Y})^2$	$\hat{u}_i^2$
49	1	47.509	1.491	125.44	94.27	2.22
45	2	45.352	-0.352	51.84	57.03	0.12
44	3	43.194	0.806	38.44	29.10	0.65
39	4	41.037	-2.037	1.44	10.48	4.15
38	5	38.879	-0.879	0.04	1.16	0.77
37	6	36.721	0.279	0.64	1.16	0.08
34	7	34.564	-0.564	14.44	10.47	0.32
33	8	32.406	0.594	23.04	29.09	0.35
30	9	30.249	-0.249	60.84	57.02	0.06
29	10	28.091	0.909	77.44	94.26	0.83
				393.6	384.06	9.55

$$r^2 = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = \frac{ESS}{TSS}$$

$$r^2 = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = \frac{384.06}{393.6} \approx 0.98$$

Approximately 98 percent of the variation in Y is explained by variation in X.

Two properties of r^2

1. Nonnegative quantity
2. Its limits are $0 \le r^2 \le 1$

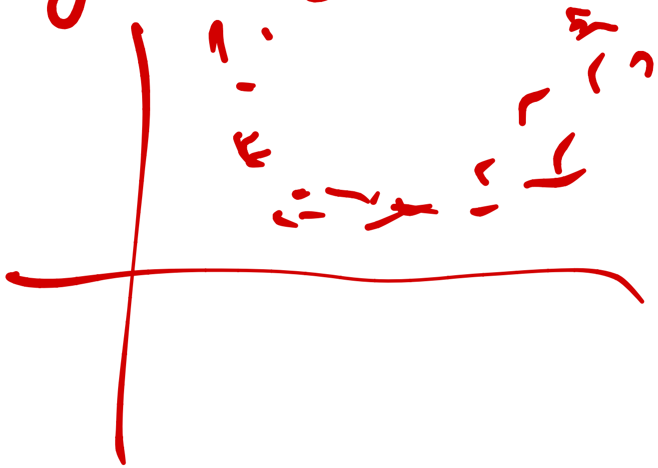
The coefficient of correlation: r

- A measure of the degree of association between two variables

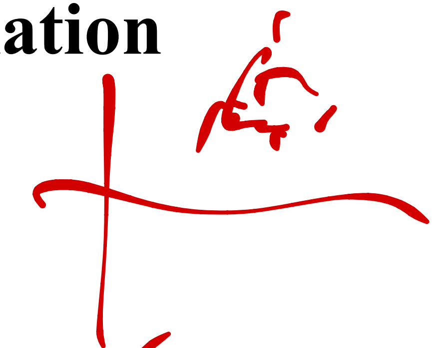
$$r = \pm\sqrt{r^2}$$

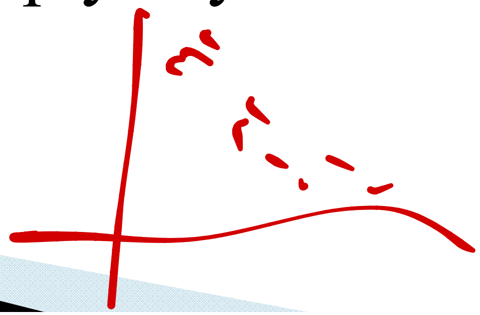
Properties of *r*

$r_{xy} = r_{yx}$

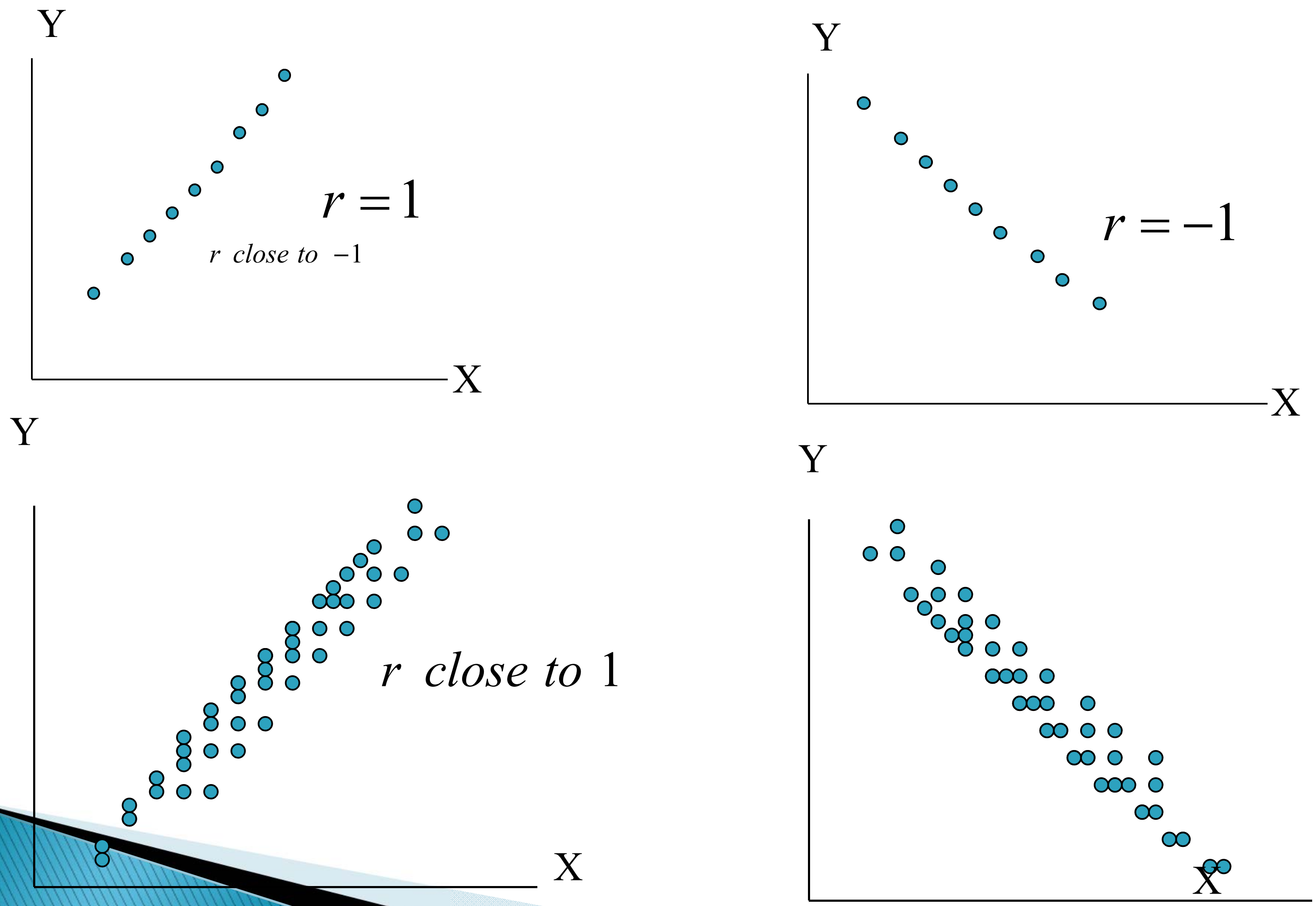


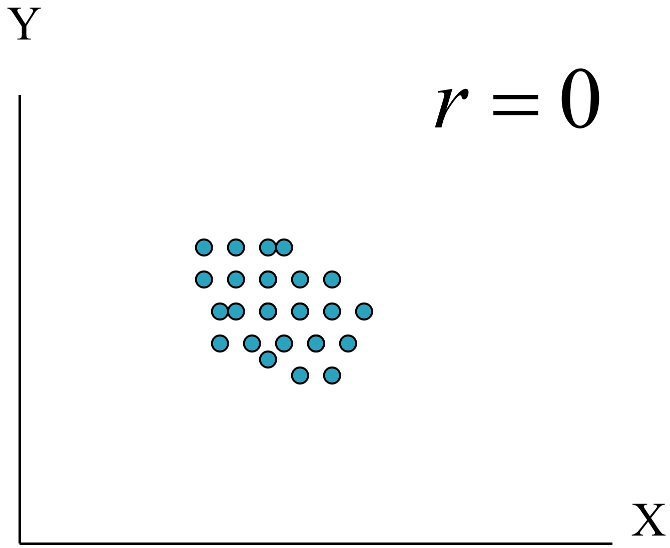
1. Can be positive or negative
2. Lies between the limits of -1 and 1
3. Symmetric in nature
4. Independent or the origin and scale
5. If X and Y are statistically independent, the correlation coefficient between them is zero **but zero correlation does not necessarily imply independence**
6. No meaning for describing nonlinear relations
7. Does not necessarily imply any cause and effect relationship

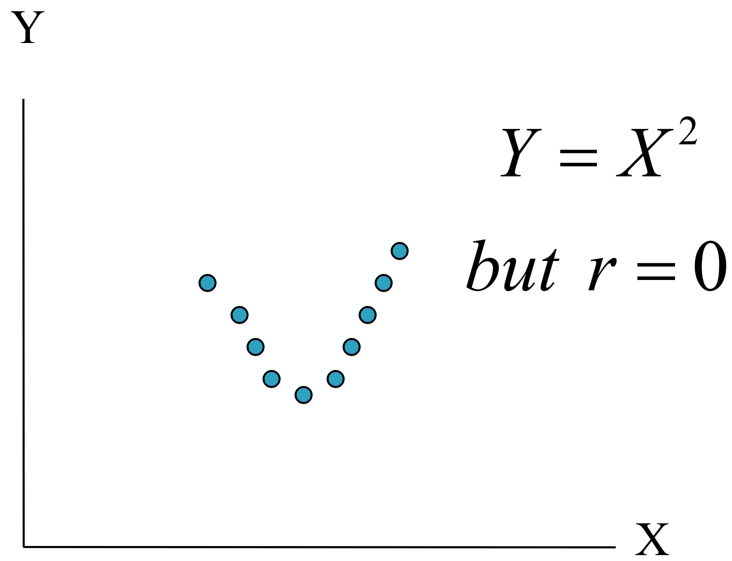




Correlation patterns




Y
$r = 0$
X


Y
$Y = X^2$
$but\ r = 0$
X

Source

Gujarati, D.N. (2009) Basic Econometrics. 5th ed. Singapore, McGraw-Hill.