EE431 Economics of Financial Markets and Institutions
Homework 4: CAPM and APT

Solution

1. Let $R_i = a_i + \beta_i R_m + \epsilon_i$ and $\sigma_i^2 = \beta\sigma_{im} + \sigma_{\epsilon_i}^2$. Use the information given. Explain the two components of risk. Which type of risk can be reduced through diversification? According to CAPM, what is the measure of risk for individual asset i?

   ANSWER.

   "In equilibrium, every asset must be priced so that its risk-adjusted rate of return falls exactly in the SML. Its expected rate of return can be described by CAPM, $E(R_i) = R_f + \beta_i[E(R_m) - R_f]$. However, all assets fall on CML. This means not all the vairance on asset's return is of concern to risk-averse investors. Investors can diversify away all risk except the **covaiance of an asset with the market portfolio**. In other words, they can diversify away all risk except the risk of the economy as a whole, which is onescapable (undiversificable). Consequently, the only risk that investors will pay a premium to avoid is covariance risk. The total risk of any individual asset can be partitioned into two parts - ***systematic risk*, which is a measure of how asset covaires with the economy**, and ***unsystematic risk***, **which is independent of the economy**:

   total risk =systematic risk +unsystematic risk.

   Mathematical precision can be attached to this concept by nothing empirically the return on any asset is a linear function of market return plus a random error term ϵ_i, which is independednt of the market:

   $R_i = a_i + b_i R_m + \epsilon_i$.

   The equation contains three terms; a constant a_i, which has no variance; a constant times a random vairable $b_i R_m$; and a second random variable ϵ_i, which has zero covariance with R_m.

   $\sigma_i^2 = E[R_i - ER_i]^2$

   $ER_i = a_i + b_i E(R_m)$

$$
\begin{aligned}
\sigma_i^2 &= E[R_i - ER_i]^2 \\
&= E[a_i + b_i R_m + \epsilon_i - (a_i + b_i E(R_m))]^2 \\
&= E[a_i + b_i R_m + \epsilon_i - a_i - b_i E(R_m)]^2 \\
&= E[b_i(R_m - E(R_m)) + \epsilon_i]^2 \\
&= b_i^2 E[(R_m - E(R_m)]^2 + b_i E[(R_m - E(R_m))(\epsilon_i)] + E[\epsilon_i^2] \\
&= b_i^2\sigma_m^2 + 0 + \sigma_\epsilon^2 \\
&= \underbrace{b_i^2\sigma_m^2}_{\text{systematic risk}} + \underbrace{\sigma_\epsilon^2}_{\text{unsystematic risk}} \\
&= \underbrace{b_i\sigma_{im}}_{\text{systematic risk}} + \underbrace{\sigma_\epsilon^2}_{\text{unsystematic risk}}
\end{aligned}
$$

   The variance is total risk. It can be partitioned into systematic risk, $b_i^2\sigma_m^2$ or $b_i\sigma_{im}$, and unsystematick risk, σ_ϵ^2. It turns out that b_i in the simple linear relationship bewteen individual asset return and market return is exactly the same as β_i in the CAPM. From Classical linear regression model, the slope coefficient, b_i equals $\dfrac{COV(R_i, R_m)}{VAR(R_m)} = \dfrac{\sigma_{im}}{\sigma_m^2}$, [study EE325, for the proof]. Hence, $b_i^2\sigma_m^2 = \dfrac{\sigma_{im}}{\sigma_m^2} \times b_i \times \sigma_m^2 = b_i\sigma_{im}$. **The systematic risk is the only type of risk that investors will pay to avoid. The appropriate measure of risk for a single asset is beta, its covariance with the market divided by the variance of the market. [Alternatively, the appropriate measure of risk for a single asset is its covariance with the market.]"**

2. Investors have mean-variance preferences and seek to maximize their utility. There are many risky assets available in the financial market. There is one risk-free asset. Assume that all assumptions of CAPM hold.

   - asset A has $\sigma_a = 0.05$;

- asset B has $\sigma_b = 0.08$
- market porfolio M has $\sigma_m = 0.10$

(a) Since investors are compensated for holding risk and $\sigma_b > \sigma_a$, is it true that expected return on security B should be greater than expected rate of return on security A at equilibrium?

ANSWER.

It is not necessary true that expected return on security B should be greater than expected rate of return on security A at equilibrium. **The standard deviation is not a good measure of risk for an individual asset. An individual asset is not lie on CML.** This means not all the vairance on asset's return is of concern to risk-averse investors. Investors can diversify away all risk except the **covariance of an asset with the market portfolio**. In equilibrium, every asset must be priced so that its risk-adjusted rate of return falls exactly in the SML. Its expected rate of return can be described by CAPM, $E(R_i) = R_f + \beta_i[E(R_m) - R_f]$. **The risk of an individual asset, A and B, should be measured by** $\beta_i = \dfrac{COV(R_i, R_m)}{\sigma_m^2}$.

To compare the risk of the two securities we must compare their beta. **Higher beta, higher expected rate of return.**

- Security B has higher standard deviation than security A, but it may have lower β than security A [$\sigma_b > \sigma_a$ but $\beta_b < \beta_a$]. In this case, according to CAPM, expected rate of return on security B would be lower than expected rate of return of security A, at equilibrium.
- Security B has higher standard deviation than security A, but it may have equal β than security A [$\sigma_b > \sigma_a$ but $\beta_b = \beta_a$]. In this case, according to CAPM, expected rate of return on security B would be equal to expected rate of return of security A, at equilibrium.
- Security B has higher standard deviation than security A, and it may have higher β than security A [$\sigma_b > \sigma_a$ but $\beta_b > \beta_a$]. In this case, according to CAPM, expected rate of return on security B would be higher than expected rate of return of security A, at equilibrium.

Then, the statement is not necessary true.

(b) r_{ij} = correlation coefficient between return on asset i and return on asset j. Let us have more information as follows.

- asset A has $r_{am} = 0.9$, $\sigma_a = 0.05$;
- asset B has $r_{bm} = 0.01$, $\sigma_b = 0.08$
- market porfolio M has $\sigma_m = 0.10$, $E(R_m) = 0.13$
- risk free rate is 1%

i. Write down CAPM equation.

ANSWER. The CAPM equation is $ER_i = R_f + [E(R_m) - R_f]\beta_i$.

- $E(R_m) = 0.13$
- $R_f = 0.01$.
- $ER_i = 0.01 + [0.13 - 0.01]\beta_i = 0.01 + 0.12\beta_i$.

ii. Calculate β_a and β_b.

ANSWER. $\beta_i = \dfrac{COV(R_i, R_m)}{\sigma_m^2}$

$$\beta_i = \frac{COV(R_i, R_m)}{\sigma_m^2} = \frac{r_{im}\sigma_i\sigma_m}{\sigma_m^2} = \frac{r_{im}\sigma_i}{\sigma_m}$$

$$\beta_a = \frac{COV(R_a, R_m)}{\sigma_m^2} = \frac{r_{am}\sigma_a\sigma_m}{\sigma_m^2} = \frac{r_{am}\sigma_a}{\sigma_m} = \frac{0.9 \times 0.05}{0.10} = 0.45.$$

$$\beta_b = \frac{COV(R_b, R_m)}{\sigma_m^2} = \frac{r_{bm}\sigma_b\sigma_m}{\sigma_m^2} = \frac{r_{bm}\sigma_b}{\sigma_m} = \frac{0.01 \times 0.08}{0.10} = 0.008.$$

iii. What are expected rates of return on asset A and asset B in equilibrium?

ANSWER. In equilibrium, expected rates of return on asset A and asset B can be described by CAPM. From (i), $ER_i = 0.01 + [0.13 - 0.01]\beta_i$.

- $ER_a = 0.01 + 0.12\beta_a$. Substitute $\beta_a = 0.45$ (from ii). $ER_a = 0.01 + 0.12 \times 0.45 = 0.064 = 6.4\%$.
- $ER_b = 0.01 + 0.12\beta_b$. Substitute $\beta_b = 0.008$ (from ii). $ER_a = 0.01 + 0.12 \times 0.008 = 0.01096 = 1.096\%$.

3. $ER_A = 0.05 + 0.5F_1$; asset A has $\beta_{A1} = 0.5$.

   $ER_B = 0.02 - 0.25F_1$; asset B has $\beta_{B1} = -0.25$.

   Notation: β_{j1} and β_{j1} for j = A; B denote the responses of the rates of return on assets A and B to the factor 1 (F_1).

   (a) Determine the portfolio weights you need to place on A and B in order to construct a portfolio which has zero exposure to the factor 1 (F_1) (a risk-free portfolio). What is the expected return of the risk-free portfolio?

   ANSWER.

   Let w_A be the weight of asset A and w_B be the weight of asset B in a portfolio. Then, expected return of the portfolio can be written as follows.

   $$\begin{aligned} E(R_p) &= aE(X) + bE(Y), \\ E(R_p) &= w_A E(R_A) + w_B E(R_B), \\ &= w_A(0.05 + 0.5F_1) + w_B(0.02 - 0.25F_1), \\ &= (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1. \end{aligned}$$

   A riskless portfolio must have beta equal to zero. Therefore,

   $0.5w_A - 0.25w_B = 0 \Rightarrow$ equation (1).

   The sum of the weights must always be 1.

   $w_A + w_B = 1 \Rightarrow$ equation (2),

   Solving the system of equations (equation (1) and equation (2)) for w_A and w_B and get $w_A = \frac{1}{3}$ and $w_B = \frac{2}{3}$.

   Substitute $w_A = \frac{1}{3}$ and $w_B = \frac{2}{3}$ in to $E(R_p) = (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1$ and get $E(R_p) = \frac{0.05 + 2(0.04)}{3} = 0.03$.

   Hence, to construct a riskless portfolio, the weight placed on asset A must be equal to $\frac{1}{3}$ and the weight placed on asset B must be equal to $\frac{2}{3}$. The return of the risk-free portfolio is then equal to 0.03 or 3%.

   (b) From (a), is there any abritrage opportunity exist if the market risk-free interest rate is 1%? If yes, explain how an investor can make an arbitrage profit.

   ANSWER.

   The risk-free portfolio in (a) and the risk-free asset avaiable in the market (which pays the market risk-free interest rate) are both riskless. Their returns are certain, regardless of the value of F_1. They have the same value of beta. They should yield the same rate of return. Otherwise, everyone would want to buy the security or the portfolio with the lower price (higher yields) and to sell the security or the portfolio with the higher price (lower yields). The rate of return of the risk-free asset porfolio in (a) is equal to 0.03. The rate of return of the risk-free asset porfolio in (a) is *higher* than tje market risk-free interest rate (for example, the interest rate on treasury bill). 3% > 1%. 0.03 > 0.01. We should *short-sell* the one that *pays the lower rate of retur*n, the risk-free asset available in the market. In other words, we borrow from the market and pay the market risk-free interest rate. After that, we use the proceeds to *buy* the one that *pays the higher rate of return, risk-free portfolio in (a)*. We get 3% rate of return from investing in a risk-free portfolio in (a). We would realize a profit of (3% − 1%) = 2%. This arbitrage opportunity is inconsistent with market equilibrium.

(c) Determine the portfolio weights you need to place on A and B in order to construct a portfolio which has a unit exposure to the factor 1 (F_1). What is the expected return of the portfolio which has a unit exposure to the factor 1?

ANSWER.

Let w_A be the weight of asset A and w_B be the weight of asset B in a portfolio. Then, expected return of the portfolio can be written as follows.

$$\begin{aligned}
E(R_p) &= aE(X) + bE(Y), \\
E(R_p) &= w_A E(R_A) + w_B E(R_B), \\
&= w_A(0.05 + 0.5F_1) + w_B(0.02 - 0.25F_1), \\
&= (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1.
\end{aligned}$$

Given that the portfolio must have beta equal to one. Therefore,

$0.5w_A - 0.25w_B = 1 \Rightarrow$ equation (1).

The sum of the weights must always be 1.

$w_A + w_B = 1 \Rightarrow$ equation (2),

Solving the system of equations (equation (1) and equation (2)) for w_A and w_B and get $w_A = \frac{5}{3}$ and $w_B = -\frac{2}{3}$.

Substitute $w_A = \frac{5}{3}$ and $w_B = -\frac{2}{3}$ in to $E(R_p) = (0.05w_A + 0.02w_B) + (0.5w_A - 0.25w_B)F_1$ and get $E(R_p) = \left(\frac{0.05(5) + (0.02)(-2)}{3}\right) + \left(\frac{0.5(5) - 0.25(-2)}{3}\right) F_1 = 0.07 + F_1$.

Hence, to construct a portfolio which has a unit exposure to the factor 1 (F_1), the weight placed on asset A must be equal to $\frac{5}{3}$ and the weight placed on asset B must be equal to $-\frac{2}{3}$ (we have to short sell asset B.). The expected return of the portfolio which has a unit exposure to the factor 1 (F_1) is then equal to $0.07 + F_1$.

(d) Suppose $ER_A = 0.05 + 0.5F_1$ and $ER_B = 0.02 - 0.25F_1$ hold true at equilibrium. Derive an equation for the equilibrium expected rate of return of an asset K which has $\beta_{K1} = b_K$. Write down the general equation.

ANSWER.

Making use $ER_i = R_f + (\lambda_1 - R_f)\beta_{i1}$, $R_f = 0.03$ (from question (a) and $E(R_p) = \lambda_1 = 0.07 + F_1$ (from question (c) , and get

$$\begin{aligned}
E(R_i) &= 0.03 + ((0.07 + F_1) - 0.03)\,\beta_{i1}, \\
&= 0.03 + (0.04 + F_1)\,\beta_{i1}, \\
&= (0.03 + 0.04\beta_{i1}) + \beta_{i1}F_1.
\end{aligned}$$

The equilibrium expected rate of return of the asset K is equal to $(0.03 + 0.04b_K) + b_K F_1$.

4. Suppose the following equations hold true at equilibrium,

$ER_A = 0.02$

$ER_B = 0.15 + F_1 + 2F_2$

$ER_{\mathrm{C}} = 0.07 + F_2$

(a) Determine the portfolio weights you need to place on A, B and C in order to construct a portfolio which has a unit exposure to the factor 1 (F_1) and zero exposure to factor 2 ; $\beta_1 = 1$ and $\beta_2 = 0$.

$E(R_p) = w_a E(R_a) + w_b E(R_b) + w_c E(R_c)$

ANSWER.

$$\begin{aligned}
E(R_p) &= w_a[0.02] + w_b[0.15 + F_1 + 2F_2] + w_c[0.07 + F_2] \\
&= [0.02w_a + 0.15w_b + 0.07w_c] + [w_b F_1] + [(2w_b + w_c)F_2]
\end{aligned}$$

We would like to construc a portfolio which has a unit exposure to the factor 1 (F_1) and zero exposure to factor 2 ; $\beta_1 = 1$ and $\beta_2 = 0$.

This means that coefficient of F_1 must be equal to 1 and coefficient of F_2 must be equal to 0.

$$\begin{aligned} w_b &= 1 \\ 2w_b + w_c &= 0 \\ w_a + w_b + w_c &= 1 \end{aligned}$$

Solve the system of these three equations.

$$\begin{aligned} w_b &= 1 \\ w_c &= -2 \\ w_a &= 2 \end{aligned}$$

Substitute w_a, w_b, w_c into the return on portfolio equation.

$$\begin{aligned} E(R_p) &= [0.02w_a + 0.15w_b + 0.07w_c] + [w_b F_1] + [(2w_b + w_c)F_2] \\ &= [0.02(2) + 0.15(1) + 0.07(-2)] + [(1)F_1] + [(2(1) + (-2))F_2] \\ &= [0.04 + 0.15 - 0.14] + F_1 \\ &= 0.05 + F_1 \end{aligned}$$

(b) From question (a). What is the expected return of the portfolio which has $\beta_1 = 1$ and $\beta_2 = 0$? i.e.$E(R_1) = \alpha_1 + F_1$.

ANSWER. Expected rate of return on the portfolio which has $\beta_1 = 1$ and $\beta_2 = 0$ must be equal to the rate of return on portfolio in questuion (a); otherwise, there is an arbitrage opportunity.

$E(R_1) = 0.05 + F_1$.

(c) From the information given, determine the rate of return on portfolio which has $\beta_1 = 0$ and $\beta_2 = 1$. i.e.$E(R_2) = \alpha_2 + F_2$.

ANSWER. Expected rate of return on the portfolio which has $\beta_1 = 0$ and $\beta_2 = 1$ must be the same as ER_C.

$E(R_2) = 0.07 + F_2$

(d) From the information given, determine the rate of return on portfolio which has $\beta_1 = 0$ and $\beta_2 = 0$. i.e.$R_0 = \alpha_0$.

ANSWER. Expected rate of return on the portfolio which has $\beta_1 = 0$ and $\beta_2 = 0$ must be the same as ER_A.

$R_0 = 0.02$

(e) From the information given [question (b), (c), (d)], write down the general equation for Arbitrage Pricing Model. [Hint: $E(R_i) = R_0 + (E(R_1) - R_0)\beta_1^i + (E(R_2) - R_0)\beta_2^i$]

ANSWER. $E(R_i) = R_0 + (E(R_1) - R_0)\beta_1^i + (E(R_2) - R_0)\beta_2^i$

Substitute $R_0 = 0.02$, $E(R_1) = 0.05 + F_1$ and $E(R_2) = 0.07 + F_2$.

$$\begin{aligned} E(R_i) &= R_0 + (E(R_1) - R_0)\beta_1^i + (E(R_2) - R_0)\beta_2^i \\ &= 0.02 + [0.05 + F_1 - 0.02]\,\beta_1^i + [0.07 + F_2 - 0.02]\,\beta_2^i \\ &= [0.02 + 0.03\beta_1^i + 0.05\beta_2^i] + \beta_1^i F_1 + \beta_2^i F_2. \end{aligned}$$

Hence, the general equation for Arbitrage Pricing Model is $E(R_i) = [0.02 + 0.03\beta_1^i + 0.05\beta_2^i] + \beta_1^i F_1 + \beta_2^i F_2$.