Note Title 9/9/2016

Last time

$$y''(t) + a_1 y'(t) + a_2 y(t) = b \qquad -(1)$$

Homo. Eqn $\Rightarrow$ $y''(t) + a_1 y'(t) + a_2 y(t) = 0 \qquad -(5)$

solve for y_c

by solving a characteristic eqn $r^2 + a_1 r + a_2 = 0$

Case 1: (Two) Distinct Real Roots $a_1^2 > 4a_2$

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

$\left.\begin{matrix} r_1 < 0 \\ r_2 < 0 \end{matrix}\right\}$ Dynamic stable of Equilibrium

$y_c \to 0$ as $t \to \infty$

Case 2: (Two) Repeated Real Roots $a_1^2 = 4a_2$

$$r_1 = r_2 = r = -\frac{a_1}{2}$$

$$y_c = A_3 e^{rt} + A_4 t e^{rt}$$

$r < 0 \Rightarrow$ Dynamic stable of Equilibrium

$y_c \to 0$ as $t \to \infty$

Case 3: Complex Roots $\quad a_1^2 < 4a_2$

$$\sqrt{a_1^2-4a_2} < 0$$

imaginary numbers $\quad i = \sqrt{-1}$

$$i^2 = -1$$

$$\sqrt{-9} = \sqrt{9}\sqrt{-1}$$
$$= 3i$$
$$\sqrt{-2} = \sqrt{2}\sqrt{-1}$$
$$= (\sqrt{2})i$$

$$\sqrt{a_1^2-4a_2} = \sqrt{(4a_2-a_1^2)(-1)}$$

$$= (\sqrt{4a_2-a_1^2})i$$

$$\therefore \quad r_1 = -\frac{a_1}{2} + \frac{\sqrt{4a_2-a_1^2}}{2} i$$

$$r_2 = -\frac{a_1}{2} - \frac{\sqrt{4a_2-a_1^2}}{2} i$$

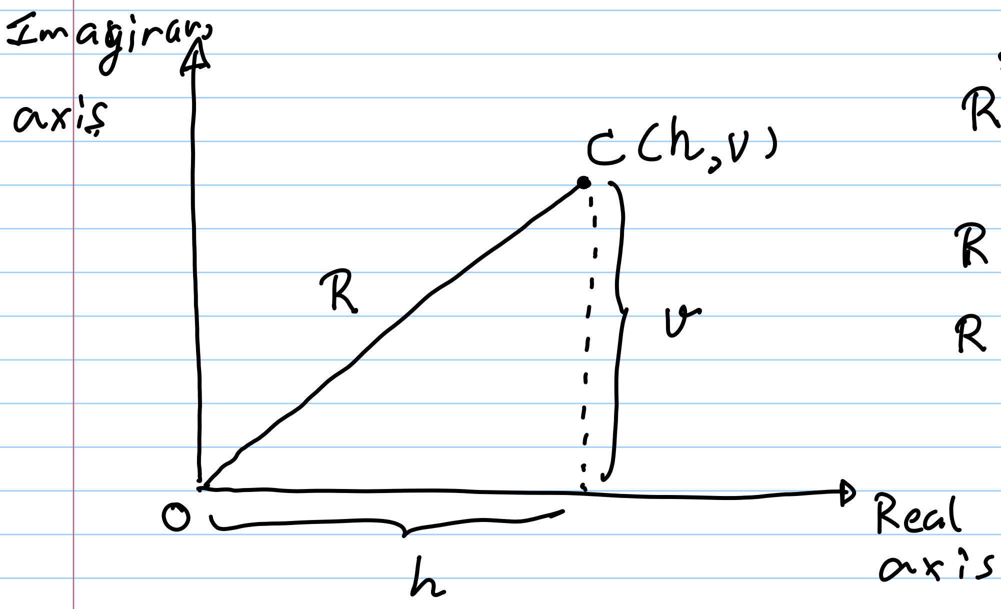
$\underline{h} + \underline{vi} \Rightarrow$ a complex number

a real part $\qquad$ an imaginary part

$h, v \Rightarrow$ real numbers

we can draw the complex number $h+vi$

graphically using an ARGAND Diagram.



$$R^2 = h^2 + v^2$$

R is always ⊕

R is called the modulus of (absolute value) the complex number $h + vi$

ⓒ Complex Roots

$$r_1, r_2 = h \pm vi$$

$$h = -\frac{a_1}{2} \qquad v = \sqrt{4a_2 - a_1^2}$$

a pair of "conjugate" complex numbers

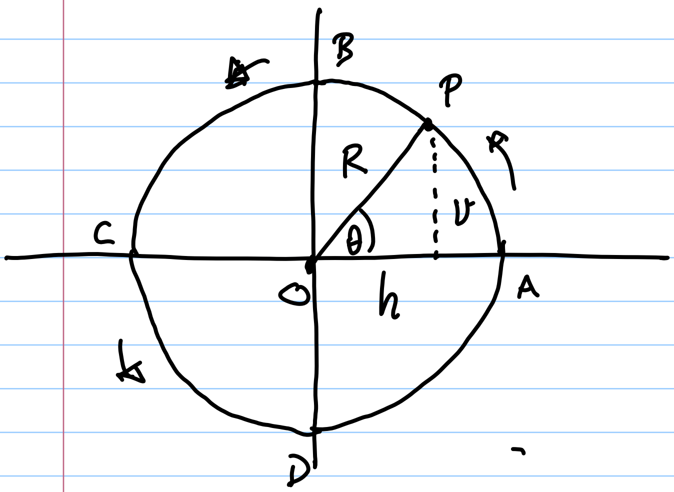
both roots share the same modulus R

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

$$= A_1 e^{(h+vi)t} + A_2 e^{(h-vi)t}$$

$$\boxed{y_c = e^{ht}(A_1 e^{vit} + A_2 e^{-vit})} \quad \text{— (10)}$$

© A Circular function



radius rotates

R does not change

h, v } Δ

$\frac{v}{R}$ and $\frac{h}{R}$ ⇒ Δ with θ

$$\frac{v}{R} = \sin\theta \Rightarrow v = R\sin\theta$$

$$\frac{h}{R} = \cos\theta \Rightarrow h = R\cos\theta$$

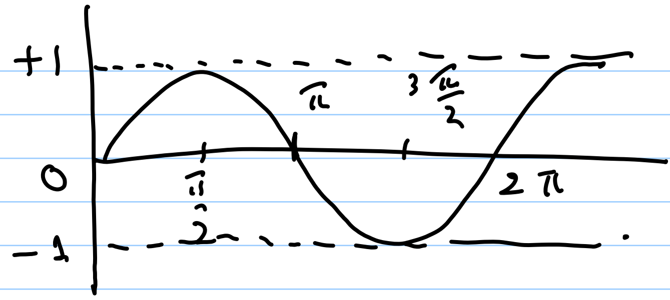
circular functions

$$R^2 = \sqrt{h^2 + v^2}$$

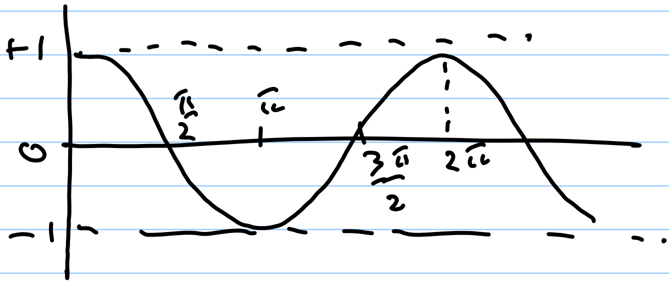
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{v}{h} \quad (h \neq 0)$$

(1) Degrees (θ)	0	90	180	270	360
(2) radian	0	$\frac{\pi}{2}$	π	$3\frac{\pi}{2}$	2π
$\frac{v}{R} = \sin\theta$	0	1	0	−1	0
$\frac{h}{R} = \cos\theta$	1	0	−1	0	1

$\sin\theta$



$\cos\theta$



NOTES

1) $\sin(\theta + 2n\pi) = \sin\theta$

$\cos(\theta + 2n\pi) = \cos\theta$

for any integer n

2) $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

3) $\sin^2\theta + \cos^2\theta = 1$ [$\sin^2\theta = (\sin\theta)^2$

4) $\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$

$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$

5) $\frac{d\sin\theta}{d\theta} = \cos\theta \Rightarrow \frac{d^2\sin\theta}{d\theta} = -\sin\theta$

$\frac{d\cos\theta}{d\theta} = -\sin\theta \Rightarrow \frac{d^2\cos\theta}{d\theta} = -\cos\theta$

(c) Euler Relation

$$h + vi = R\cos\theta + (R\sin\theta)\,i$$

$$= R(\cos\theta + i.\sin\theta) \qquad — (11)$$

$$\therefore\; h - vi = R(\cos\theta - i\sin\theta) \qquad — (12)$$

Euler Relation

$$\cos\theta + i\sin\theta = e^{i\theta} \qquad — (13)$$

$$\cos\theta - i\sin\theta = e^{-i\theta} \qquad — (14)$$

(11) becomes

$$h + vi = Re^{i\theta} \qquad — (15)$$

$$h - vi = Re^{-i\theta} \qquad — (16)$$

Ex $Re^{-i\frac{\pi}{2}}$

$R = 1 \qquad \theta = -\frac{\pi}{2}$

$$h = R\cos\theta \Rightarrow h = 1\cos\left(-\frac{\pi}{2}\right)$$

$$= 1.0$$

$$= 0$$

$$v = R\sin\theta \Rightarrow v = 1\sin\left(-\frac{\pi}{2}\right)$$

$$= 1(-1)$$

$$= -1$$

$$\therefore \quad h + vi = 0 - (1)i = -i$$

$$h - vi = 0 - (-1)i = i$$

Ex $5e^{\frac{3\pi}{2}i}$

$$h + vi = -5i$$

$$h - vi = 5i$$

Recall $$y_c = e^{ht}\left(A_1 e^{vit} + A_2 e^{-vit}\right) \quad \text{— (10)}$$

Let $\theta = vt$

From Euler Relation $$e^{vit} = e^{\theta i}$$

$$= (\cos\theta + i\sin\theta)$$

$$= (\cos vt + i\sin vt)$$

$$\therefore \quad e^{-vit} = e^{-\theta i}$$

$$= (\cos vt - i\sin vt)$$

$$y_c = e^{ht}\left[A_1(\cos vt + i\sin vt) + A_2(\cos v_t - i\sin vt)\right]$$

$$= e^{ht}\left[(A_1 + A_2)\cos vt + (A_1 - A_2)i\sin vt\right]$$

$$\boxed{y_c = e^{ht}\left[A_5 \cos vt + A_6 \sin vt\right]} \quad -(17)$$

where $\boxed{A_5} = A_1 + A_2$

$\boxed{A_6} = (A_1 - A_2)i$ //

For the complex Roots

$$y_c = e^{ht}(A_1 e^{vit} + A_2 e^{-vit})$$

$$y_c = e^{ht}(A_5 \cos vt + A_6 \sin vt)$$

$$h = -\frac{a_1}{2} \qquad v = \frac{\sqrt{4a_1 - a_2^2}}{2}$$

Ex

$$y''(t) + 2y'(t) + 17y(t) = 34$$

Find the general soln

$a_1 = 2 \qquad a_2 = 17 \qquad b = 34$

(c) Find y_p

$$\left.\begin{aligned} y_p &= k \\ y'(t) &= 0 \\ y''(t) &= 0 \end{aligned}\right\} \text{ sub into the original DE}$$

$$0 + 2(0) + 17(k) = 34$$

$$k = \frac{34}{17}$$

$$= 2.$$

The particular integral $y_p = 2$

(d) Find y_c

solving $\quad r^2 + 2r + 17 = 0$

$$r_1, r_2 = \frac{-2 \pm \sqrt{4 - 68}}{2}$$

$$= -1 \pm \frac{\sqrt{-64}}{2}$$

The characteristic roots are the pair of conjugate complex number $h + vi$

$h \pm vi$ $\qquad h = -\frac{a_1}{2} = -1$

$$v = \frac{\sqrt{4a_2 - a_1^2}}{2} = \frac{\sqrt{64}}{2} = 4$$

$$y_c = e^{ht}(A_5 \cos vt + A_6 \sin vt)$$

$$= e^{-t}(A_5 \cos 4t + A_6 \sin 4t)$$

The general sol^{n} is

$$y(t) = y_c + y_p$$

$$= e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 2$$

Let $y(0) = 3$ and $y'(0) = 11$

Find the definite sol^{n}

At $t=0$

$$y(0) = e^0\left[A_5 \cos 4(0) + A_6 \sin 4(0)\right] + 2$$

$$3 = \left[A_5 \cos 0 + A_6 \sin 0\right] + 2$$

$$3 = A_5(1) + A_6(0) + 2$$

$$3 = A_5 + 2$$

$$\boxed{A_5 = 1}$$

$$y'(t) = \left[(-1)e^{-t}(A_5\cos 4t + A_6\sin 4t)\right]$$

$$+ e^{-t}\frac{\partial}{\partial\theta}(A_5\cos 4t + A_6\sin 4t)$$

$$= \left[-e^{-t}(A_5\cos 4t + A_6\sin 4t)\right]$$

$$+ e^{-t}\left[A_5 4(-\sin 4t) + A_6 4(\cos 4t)\right]$$

$$y'(t) = -e^{-t}A_5\cos 4t - e^{-t}A_6\sin 4t - 4e^{-t}A_5\sin 4t$$

$$+ 4e^{-t}A_6\cos 4t$$

at $t=0$

$$y'(0) = -e^{0}A_5\cos(0) - e^{0}A_6\sin(0) - 4e^{0}A_5\sin(0)$$

$$+ 4e^{0}A_6\cos(0)$$

$$11 = -A_5(1) + 4A_6(1)$$

$$11 = -1 + 4A_6$$

$$A_6 = 3$$

The Definite sol^n is

$$y(t) = e^{-t}[\cos 4t + 3\sin 4t] + 2$$