Exercise: Curve Sketching

1. Find all *vertical*, *horizontal*, *slant* asymptotes (if any) of the following relations.

   (a)
   $$r = \left\{(x,y) \in X \times Y \middle| \quad yx^3 + y = -2x^3 - 3x + 5\right\}.$$

   (b)
   $$s = \left\{(x,y) \in X \times Y \middle| \quad 2x^2 + x - xy - y = 5\right\}.$$

   (c)
   $$f = \left\{(x,y) \in X \times Y \middle| \quad y = \frac{6x^3 - 1}{18 - 2x^2}\right\}.$$

2. Consider the relation f defined from X to Y, $X, Y \subseteq \mathbb{R}$,
   $$f = \left\{(x,y) \in X \times Y \middle| \quad y = \frac{x^3}{x^2 - 4}\right\}.$$

   Hint: For $f(x) = \frac{x^3}{x^2-4}$, the first and the second derivatives are given, respectively, by
   $$f'(x) = \frac{x^4 - 12x^2}{(x^2-4)^2}, \qquad \text{and} \qquad f''(x) = \frac{8x^3 + 96x}{(x^2-4)^3}.$$

   (a) Find the domain of f.
   (b) Find x-intercepts and y-intercepts (if any).
   (c) Determine the symmetry of f.
   (d) Find the horizontal and vertical asymptotes for f (if any).
   (e) Find the critical number of f. Determine the intervals on which f is increasing and decreasing. Determine the extrema (maximum and minimum) of f.
   (f) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).
   (g) Sketch the curve of f.

Optional Problems

1. Find the critical numbers of the given functions on $(-\infty, \infty)$.

   (a) $f(x) = x^3 - 3x^2 + 3x - 1$ (b) $f(x) = \frac{x^2}{x^2+2}$ (c) $f(x) = e^{-x} + 2x$
   (d) $f(x) = -x + \sin(x)$ (e) $f(x) = x^2 - 8\ln(x)$

2. Determine the intervals on which the given function f is increasing and the interval on which f is decreasing.

   (a) $f(x) = x^2 + 6x - 1$ (b) $f(x) = x^4 - 4x^3 + 9$ (c) $f(x) = x^2e^{-x}$ (d) $f(x) = x^2 - 2|x|$

3. Use the *First Derivative Test* to find the relative extrema of the given function.

   (a) $f(x) = x^3 - 3x$ (b) $f(x) = x^3 + x - 3$ (c) $\frac{x^2+3}{x+1}$

4. For each given function,
   (i) use the Second Derivative Test, when applicable, to find the relative extrema;
   (ii) find intercepts, vertical/horizontal asymptotes, and points of inflection,(if any);
   (iii) find the intervals on which it is increasing and the interval on which it is decreasing;
   (iv) find the intervals on which it is concave up and the intervals on which it is concave down; and **(v)** sketch the graph.

   (a) $f(x) = x^3 + 3x^2 + 3x + 1$ (b)$f(x) = 6x^5 - 10x^3$ (c)$f(x) = \frac{x}{x^2+2}$
   (d) $f(x) = 2x - x\ln(x)$ (e) $\frac{x^2+3}{x+1}$

5. Let $X, Y_1, Y_2 \subseteq \mathbb{R}$. Consider the relations f_1, f_2 defined by

$$f_1 = \left\{ (x, y) \in X \times Y_1 \middle| \quad y = \sqrt{\frac{x+1}{2x-3}} \right\}, \quad f_2 = \left\{ (x, y) \in X \times Y_2 \middle| (x, -y) \in f_1 \right\}.$$

   Sketch the curve of the relation $f = f_1 \cup f_2$.

6. Find the vertical and horizontal asymptotes (if any) of $f(x) = \frac{x^2-1}{2x}$ and show that $y = \frac{x}{2}$ is a slant asymptote of f.