EE320 Chapter 10
Integration and Its Application

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1 Terminology

Integration = anti-derivative or inverse of differentiation

y = F(x) $\Rightarrow$ F(x) = $\int F'(x)dx + c$

$\frac{dy}{dx} = F'(x)$

Indefinite integral

If we integrate f(x) where values of x are not given, we have to integrate without a limit. (i.e. to find indefinite integral)

A symbol for integrating a function f(x) is

$$\int f(x)dx = F(x) + c \Leftrightarrow F'(x) = f(x)$$

where

$\int$	is	integral sign
f(x)	is	integrand
c	is	constant of integration
dx	indicates	the variable involved in the integration

Note function does not have a unique integral

<u>ex.</u> Given a function $f(x) = 3x^2$, a possible result from integration of $f(x)$ is that: F(x) $= x^3 + 1$, $x^3 + 7$, $x^3 + c$

2 Basic Rules for Integration

I) $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$

II) $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$$\Rightarrow \int a^{bx} dx = \frac{1}{blna}a^{bx} + c$$

$$\Rightarrow \int f(x)e^{f(x)} dx = e^{f(x)} + c$$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = ln|f(x)| + c, f(x) \neq 0$$

III) $\int \frac{1}{x} dx = ln|x| + c$

IV) $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

V) $\int af(x) dx = a \int f(x) dx$

<u>ex.</u>

1. $\int \frac{1}{x^3} dx = -\frac{1}{2x^2} + c$

2. $\int \sqrt{x\sqrt{x\sqrt{x}}} dx = \frac{8}{15} x^{\frac{15}{8}} + c$

3. $\int (3x^4 + 5x^2 - 2) dx = \frac{3}{5} x^5 + \frac{5x^3}{3} - 2x + c$

4. $\int e^{3x} - e^{2x} + e^x dx = \frac{e^{3x}}{3} - \frac{e^{2x}}{2} + e^x + c$

5. $\int \frac{(y-2)^2}{\sqrt{y}} dy = \frac{2}{5} y^{\frac{5}{2}} - \frac{8}{3} y^{\frac{3}{2}} + 8y^{\frac{1}{2}} + c$

6. $\int 2^x dx = \frac{1}{in2} 26x + c$

VI) $\int f(u) \frac{du}{dx} dx = \int f(u) du = F(u) + c$

<u>ex.</u> $\int (ax+b)^P dx$

VII) $\int v \; du = uv - \int u \; dv$ "Integration by parts"

<u>ex.</u> $\int xe^x dx$

$$\Rightarrow \text{let } \begin{aligned} v &= x \Rightarrow dv = dx \\ u &= e^x \Rightarrow du = e^x dx \end{aligned}$$

Initial-Value Theorem

$\int f(x)dx = F(x) + c$

If we have an initial condition, we can determine the value of c.

ex. 1. Find F(x) if $F'(x) = \frac{1}{2} - 2x$ and $F(0) = \frac{1}{2}$

$\int \frac{1}{2} - 2xdx \ =$

$F(0) \ =$

2. Find F(x) if $F'(x) \ = \ x(1-x^2)$ and $\mathrm{F}(1) = \frac{5}{12}$

$\int x(1-x^2)dx \ =$

$F(0) \ =$

Application 1: Derivation of TR from MR

TR = $\int$ MR(Q) dQ

<u>ex.</u> Given MR = 10 - Q, find TR.

TR = $\int 10 - Q dQ =$

Suppose at Q = 0 , TR = 0 $\Rightarrow$ c = 0 $\therefore$ TR =

Application 2: Derivation of TC from MC

TC = $\int$ MC(Q) dQ

<u>ex.</u> Given MC = $2e^{0.2Q}$, C_F = 90, find TC.

TC(Q) =

TC(0) =

Application 3: Derivation of Profit function from MR-MC

$\Pi'(Q)$ = Marginal profit = MR-MC

$\Pi = \int \Pi'(Q) dQ$

ex. Given MR = 50 - 2Q , MC = 10+Q, find total profit when Q =10. Assume there is no fixed cost: $\Pi(10)$ = ?

$\Pi'(Q) =$

$\int \Pi'(Q)$ dQ =

If Q = 0 , TR = 0 , TC = 0 $\quad\Rightarrow\quad \Pi(Q) = 0 \quad\Rightarrow\quad c = 0$

$\therefore \Pi(Q) =$

Application 4: Derivation of Utility function from MU

U(x)= $\int MU(x)dx$

ex. Given MU(x) = $\frac{5}{3\sqrt{x}}$, find U(x).

U(x) =

Application 5: Derivation of Consumption/Saving function from marginal propensity function

ex. Given marginal propensity to save: $S'(Y) = 0.3 - 0.1Y^{-\frac{1}{2}}$ and $S(81) = 0$, find saving and consumption function

S(Y) =

S(Y) =

S(81) =

$\therefore$ S(Y) =

For C(Y), $C = Y - S =$

3 Definite Integrals

$\int_a^b f(x)\,\mathrm{d}x = F(x)\mid_a^b = F(b) - F(a)$

ex. $\int_0^2 5x^2\,\mathrm{d}x \ =$

$\int_0^1 ax^b\,\mathrm{d}x \ =$

Area and definite integral

The area under the graph of a continuous and nonnegative function f(x) over the interval [a,b] is $\int_a^b f(x)\,\mathrm{d}x$ or

$$\text{Area } A = \lim_{\Delta x \to 0} {}_a^b[f(x) - \Delta x] = \int_a^b f(x)\,\mathrm{d}x$$

Properties of Definite Integrals

1. $\int_a^b f(x)\,\mathrm{d}x = \int_b^a -f(x)\,\mathrm{d}x$
2. $\int_a^a f(x)\,\mathrm{d}x = 0$
3. $\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x\ ,\ (a<b<c)$
4. $\int_a^b -f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$
5. $\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$
6. $\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
7. $\int_{x=a}^{x=b} v\,\mathrm{d}u = uv\,|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,\mathrm{d}u$

ex.

1. $\int_0^5 (x + x^2)\mathrm{d}x \quad =$

2. $\int_2^4 x^2(\frac{1}{3}x^3 + 1)\,\mathrm{d}x \quad =$

3. $\int_{-2}^2 e^x - e^{-x}\,\mathrm{d}x \quad =$

4. $\int_e^6 (\frac{1}{x} + \frac{1}{1+x})\,\mathrm{d}x \quad =$

5. $\int_{-2}^3 |x + 1|\,\mathrm{d}x \quad =$

Application: Capital Formation and Investment Functions

Definition : K(t) = capital stock at t

$\frac{d}{dt}K(t)$ = rate of capital formation

$I(t)$ = rate of net investment flow at t

$$\frac{d}{dt}K(t) \equiv I(t) \quad \Rightarrow \quad \int I(t)dt = \int \frac{dK}{dt}dt = K(t)$$

Gross investment $\equiv I_g(t) = I(t) + \delta K(t)$

Capital formation during a time interval [a,b] $= \int_a^b I(t)\,\mathrm{d}t = K(t)\,|_a^b$

ex. Suppose net investment flow is I(t) = $3t^{\frac{1}{2}}$ and the initial capital stock at t = 0 is K = 25. What is K(t) during [1,4]?

Application: Consumer and Producer Surpluses

$\text{CS} = \int_0^{Q^*} D(Q) - P^* \,\mathrm{d}Q = \int_{P^*}^{P-\text{intercept}} D(Q)\,\mathrm{d}P$

$\text{PS} = \int_0^{Q^*} P^* - S(Q)\,\mathrm{d}Q = \int_{P-\text{intercept}}^{P^*} S(Q)\,\mathrm{d}P$

ex. S(P) $= -\frac{1}{2} + \frac{1}{2}P$

D(P) $= \frac{25}{2} - \frac{1}{2}P$

Find PS , CS and total welfare

ex. If the demand changes to D(P) $= \frac{30}{2} - \frac{1}{2}P$, ΔCS and ΔPS?

ex. If the demand does not change and government imposes $ 4 per unit tax on producer instead,

Application: First-Degree Price discrimination or Perfect Price discrimination

Monopolist charges the maximum price for each unit of output sold.

ex. Suppose that a monopolist faces a demand function P = 24 -Q, and MC = 4 + 3Q. Find CS and PS at profit-maximized Q^*

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