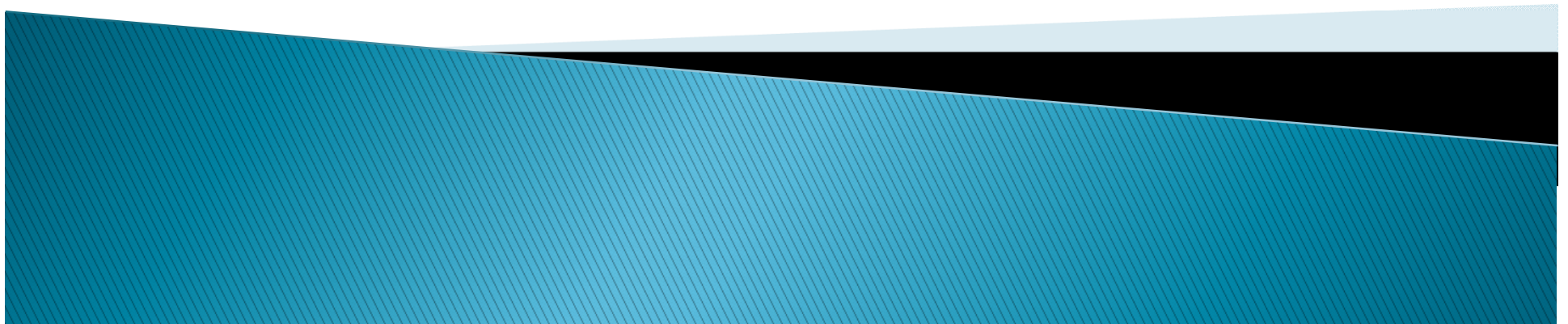


Extensions of the Two-Variable Linear Regression Model Part II



Functional Forms of Regression Models

- ▶ The log-linear model
- ▶ Semilog models
- ▶ Reciprocal models
- ▶ The logarithmic reciprocal model



The log-linear model

The exponential regression model:

$$Y_i = \beta_1 X_i^{\beta_2} e^{u_i}$$

Which may be expressed alternatively as

$$\ln Y_i = \ln \beta_1 + \beta_2 \ln X_i + u_i$$

$$\ln Y_i = \alpha_1 + \beta_2 \ln X_i + u_i$$

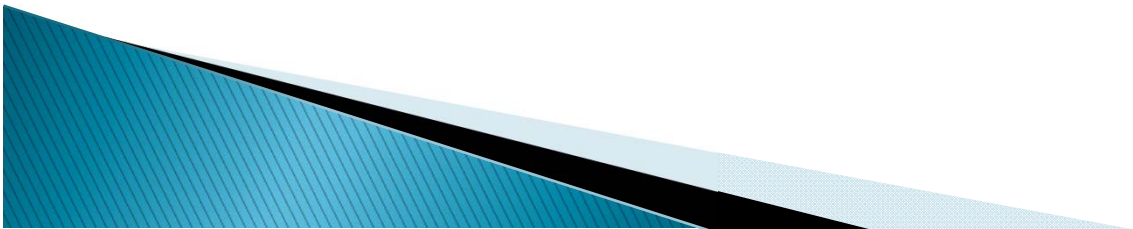
$$\text{where } \alpha = \ln \beta_1$$



$$\ln Y = \beta_1 + \beta_2 \ln X$$

$$\frac{1}{Y} \frac{dY}{dX} = \frac{\beta_2}{X} \frac{dX}{dX}$$

$$\beta_2 = \frac{X}{Y} \frac{dY}{dX} = \frac{dY/Y}{dX/X} = \textit{elasticity}$$



The log-linear model

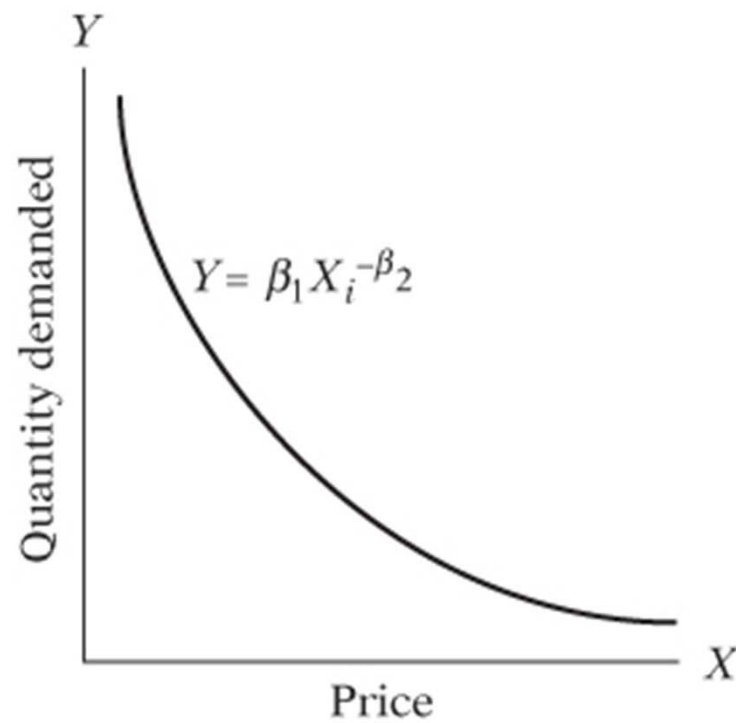
- ▶ The slope coefficient β_2 measures the elasticity of Y with respect to X, that is, the percentage change in Y for a given (small) percentage change in X



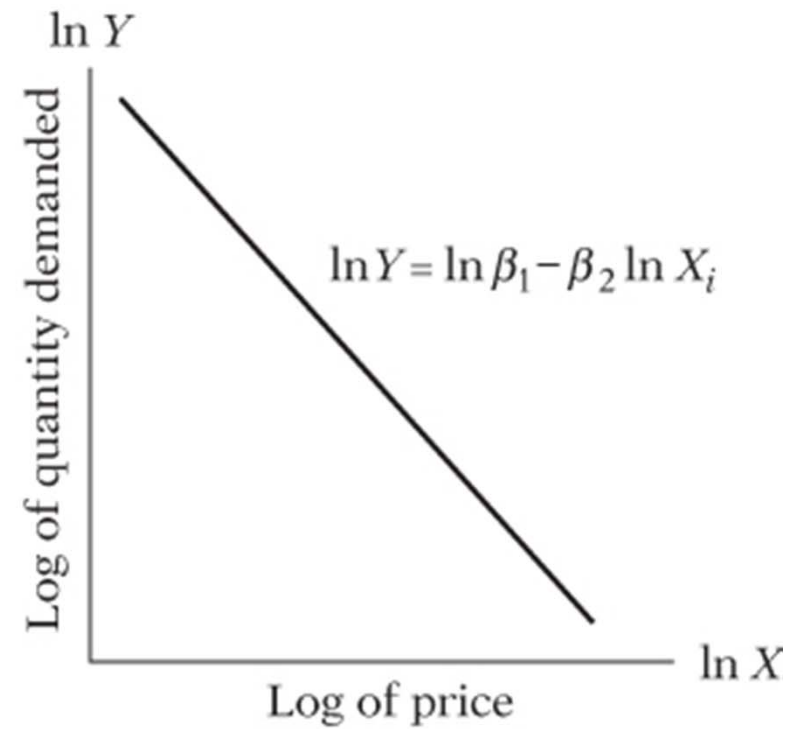
- ▶ If Y represents the quantity of a commodity demanded and X its unit price, β_2 measures the price elasticity of demand.
- ▶ If the relationship between quantity demanded and price is as shown in figure (a), the double-log transformation as shown in figure (b) will then give the estimate of the price elasticity ($-\beta_2$)



The log-linear model



(a)



(b)

Percentage change vs. Percentage point change

Example –The unemployment rate

The unemployment rate of 6%, if this rate goes to 8%, we say that **the percentage point change in the unemployment rate is 2**

The percentage change in the unemployment rate is $(8-6)/6$, or about 33%



Example

Expenditure on durable goods in relation to total personal consumption expenditure

- We wish to find the elasticity of expenditure on durable goods with respect to total personal consumption expenditure.



TABLE 6.3

**Total Personal
Expenditure and
Categories
(Billions of chained
[2000] dollars;
quarterly data at
seasonally adjusted
annual rates)**

Sources: Department of
Commerce, Bureau of
Economic Analysis.
*Economic Report
of the President, 2007,*
Table B-17, p. 347.

Year or quarter	EXPSERVICES	EXPDUR	EXPNONDUR	PCEXP
2003-I	4,143.3	971.4	2,072.5	7,184.9
2003-II	4,161.3	1,009.8	2,084.2	7,249.3
2003-III	4,190.7	1,049.6	2,123.0	7,352.9
2003-IV	4,220.2	1,051.4	2,132.5	7,394.3
2004-I	4,268.2	1,067.0	2,155.3	7,479.8
2004-II	4,308.4	1,071.4	2,164.3	7,534.4
2004-III	4,341.5	1,093.9	2,184.0	7,607.1
2004-IV	4,377.4	1,110.3	2,213.1	7,687.1
2005-I	4,395.3	1,116.8	2,241.5	7,739.4
2005-II	4,420.0	1,150.8	2,268.4	7,819.8
2005-III	4,454.5	1,175.9	2,287.6	7,895.3
2005-IV	4,476.7	1,137.9	2,309.6	7,910.2
2006-I	4,494.5	1,190.5	2,342.8	8,003.8
2006-II	4,535.4	1,190.3	2,351.1	8,055.0
2006-III	4,566.6	1,208.8	2,360.1	8,111.2

Note: See Table B-2 for data for total personal consumption expenditures for 1959–1989.

EXPSERVICES = expenditure on services, billions of 2000 dollars.

EXPDUR = expenditure on durable goods, billions of 2000 dollars.

EXPNONDUR = expenditure on nondurable goods, billions of 2000 dollars.

PCEXP = total personal consumption expenditure, billions of 2000 dollars.

(Continued)



$$\ln \widehat{\text{EXDUR}}_i = -7.5417 + 1.6266 \ln \text{PCEX}_t$$

As these result shows , the elasticity of EXPDUR with respect to PCEX is about 1.63, suggesting that **if total personal expenditure goes up by 1 percent, on average, the expenditure on durable goods goes up by about 1.63 percent**

Thus, expenditure on durable goods is very responsive to changes in personal consumption expenditure. This is one reason why producers of durable goods keep a keen eye on changes in personal income and personal consumption expenditure



Semilog Models

- ▶ Log-Lin model

$$\ln Y_i = \beta_1 + \beta_2 X_i + u_i$$

- ▶ Lin-Log model

$$Y_i = \beta_1 + \beta_2 \ln X_i + u_i$$



The Log-Lin model

$$\ln Y_i = \beta_1 + \beta_2 X_i + u_i$$



- ▶ The rate of growth of certain economic variables such as population, GNP, money supply employment, productivity, and trade deficit



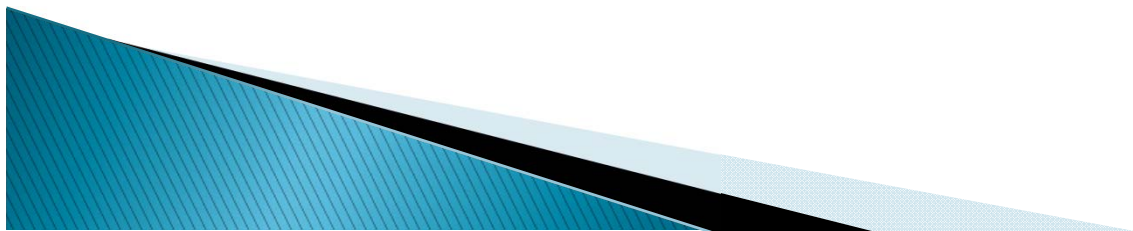
Suppose we want to find out the growth rate of personal consumption expenditure on services

Y_t denote real expenditure on services at time t and Y_0 the initial value of the expenditure on services

Recall the following well-known compound interest formula from your introductory course in economics

$$Y_t = Y_0(1 + r)^t$$

where r is the compound rate of growth of Y



Taking the natural logarithm, we can write

$$\ln Y_t = \ln Y_0 + t \ln(1 + r)$$

Now letting

$$\beta_1 = \ln Y_0$$

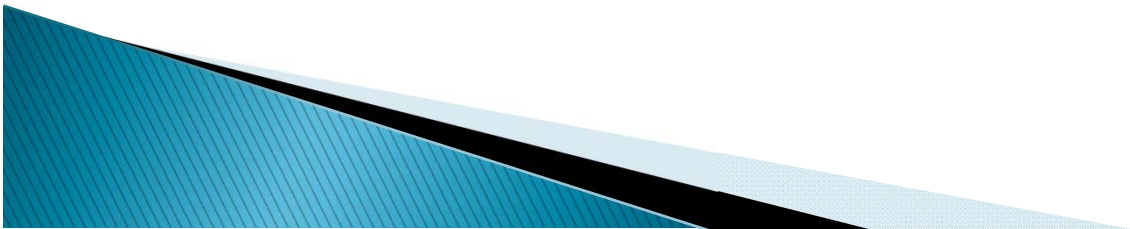
$$\beta_2 = \ln(1 + r)$$

We can write as

$$\ln Y_t = \beta_1 + \beta_2 t$$

We obtain

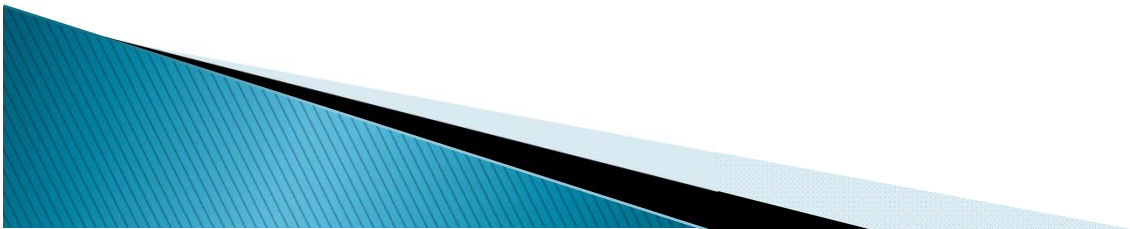
$$\ln Y_t = \beta_1 + \beta_2 t + u_t$$



$$\ln Y = \beta_1 + \beta_2 X$$

$$\frac{1}{Y} \frac{dY}{dX} = \beta_2$$

$$\beta_2 = \frac{dY/Y}{dX}$$



- ▶ Using differential calculus one can show that

$$\beta_2 = \frac{d(\ln Y)}{dX} = \frac{\left(\frac{1}{Y}\right)}{\left(\frac{dY}{dX}\right)} = \frac{\left(\frac{dY}{Y}\right)}{dX}$$

For a small changes in Y and X this relation may be

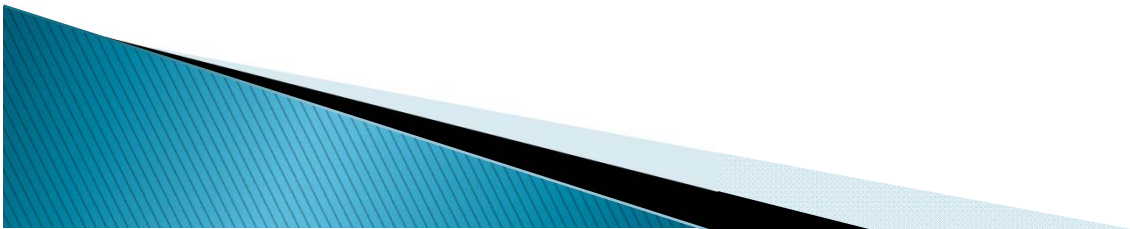
approximated by $\frac{\frac{Y_t - Y_{t-1}}{Y_{t-1}}}{(X_t - X_{t-1})}$



$$\beta_2 = \frac{\text{relative change in regressand}}{\text{absolute change in regressor}}$$

β_2 is known as the semielasticity of Y with respect to X

$100 * \beta_2$ is known as the semielasticity of Y with respect to X



Example

- ▶ We want to find out the growth rate of personal consumption expenditure on services for the data
- ▶ The regression results over time (t) are as follows:



TABLE 6.3

**Total Personal
Expenditure and
Categories
(Billions of chained
[2000] dollars;
quarterly data at
seasonally adjusted
annual rates)**

Sources: Department of
Commerce, Bureau of
Economic Analysis.
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Note: See Table B-2 for data for total personal consumption expenditures for 1959–1989.

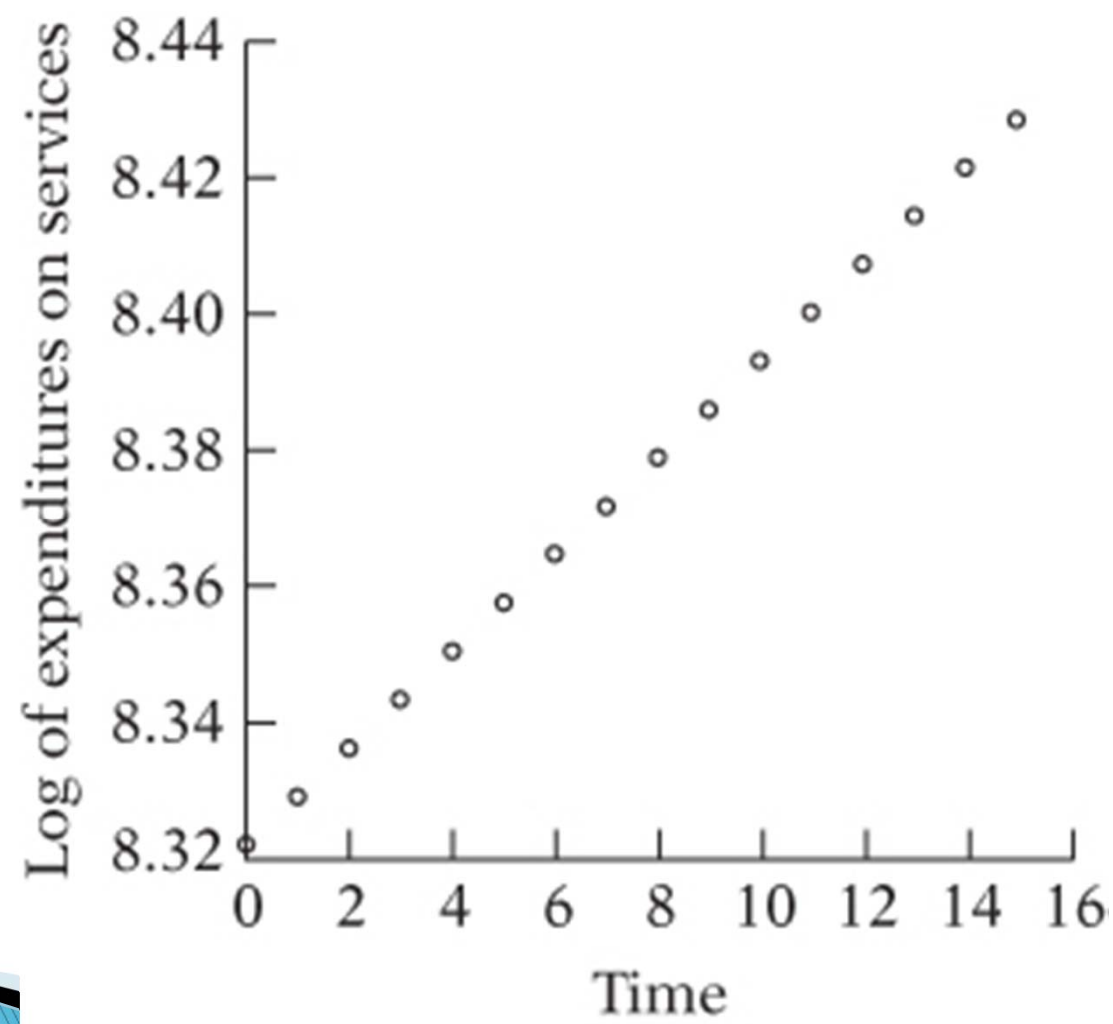
EXPSERVICES = expenditure on services, billions of 2000 dollars.

EXPDUR = expenditure on durable goods, billions of 2000 dollars.

EXPNONDUR = expenditure on nondurable goods, billions of 2000 dollars.

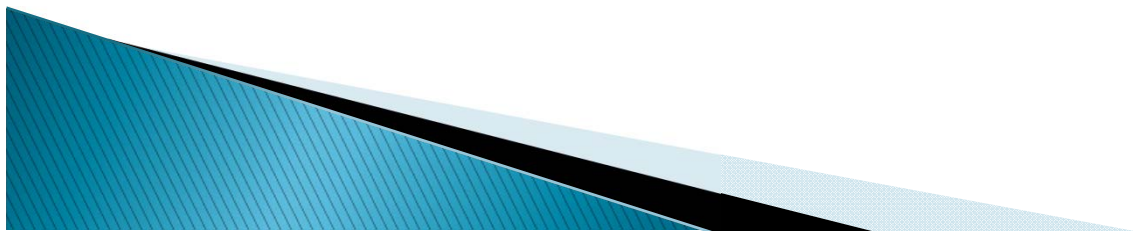
PCEXP = total personal consumption expenditure, billions of 2000 dollars.

(Continued)



$$\ln \widehat{EXS}_t = 8.3226 + 0.00705t$$

Over the quarterly period 2003-I to 2006-III, expenditures on services increased at the (quarterly) rate of 0.705 percent. Roughly, this is equal to an annual growth rate of 2.82 percent



Linear Trend Model

$$Y_t = \beta_1 + \beta_2 t + u_t$$

$$\widehat{EXS}_t = 4111.545 + 30.674t$$
$$t = (655.5628) \quad (44.4671)$$

$$r^2 = 0.9935$$

Over the quarterly period 2003-I to 2006-III, on average, expenditure on services increased at the absolute rate of about 30 billion dollars per quarter. That is, there was an upward trend in the expenditure on services



- ▶ The choice between the growth rate model and the linear trend model will depend upon whether one is interested in the relative or absolute change in the expenditure on services, although for comparative purposes it is the relative change that is generally more relevant
- ▶ We cannot compare the r^2 values of models because the regressands in the two models are different



Lin-Log models

The absolute change in Y for a percentage change in X

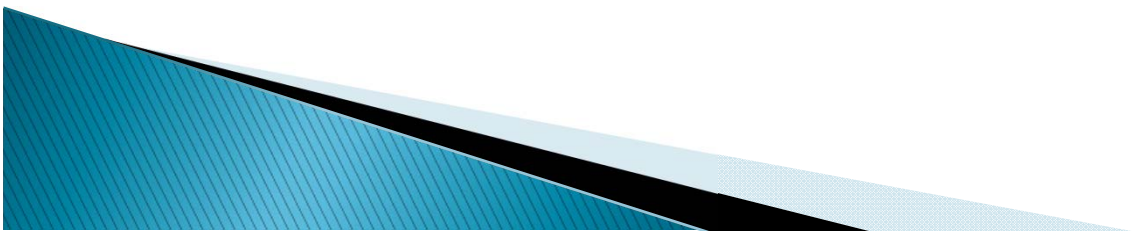
$$Y_i = \beta_1 + \beta_2 \ln X_i + u_i$$



$$Y = \beta_1 + \beta_2 \ln X$$

$$\frac{dY}{dX} = \beta_2 \frac{1}{X}$$

$$\beta_2 = \frac{dY}{dX / X}$$



- ▶ Using differential calculus one can show that

$$\frac{dY}{dX} = \beta_2 \left(\frac{1}{X} \right)$$

Therefore,

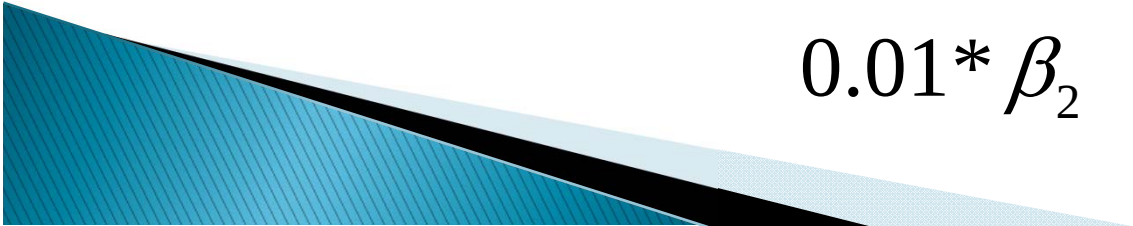
$$\beta_2 = \frac{dY}{dX} \frac{X}{Y}$$



$$\beta_2 = \frac{\text{Change in } Y}{\text{relative change in } X}$$

The absolute change in Y for a percentage change in X

If $\Delta X / X$ changes by 0.01 unit or 1%, the absolute change in Y is

$$0.01 * \beta_2$$


Engel expenditure models

Engel postulated that “the total expenditure that is devoted to food tends to increase in arithmetic progression as total expenditure increases in geometric progression”



Example

- ▶ Food expenditure in India example

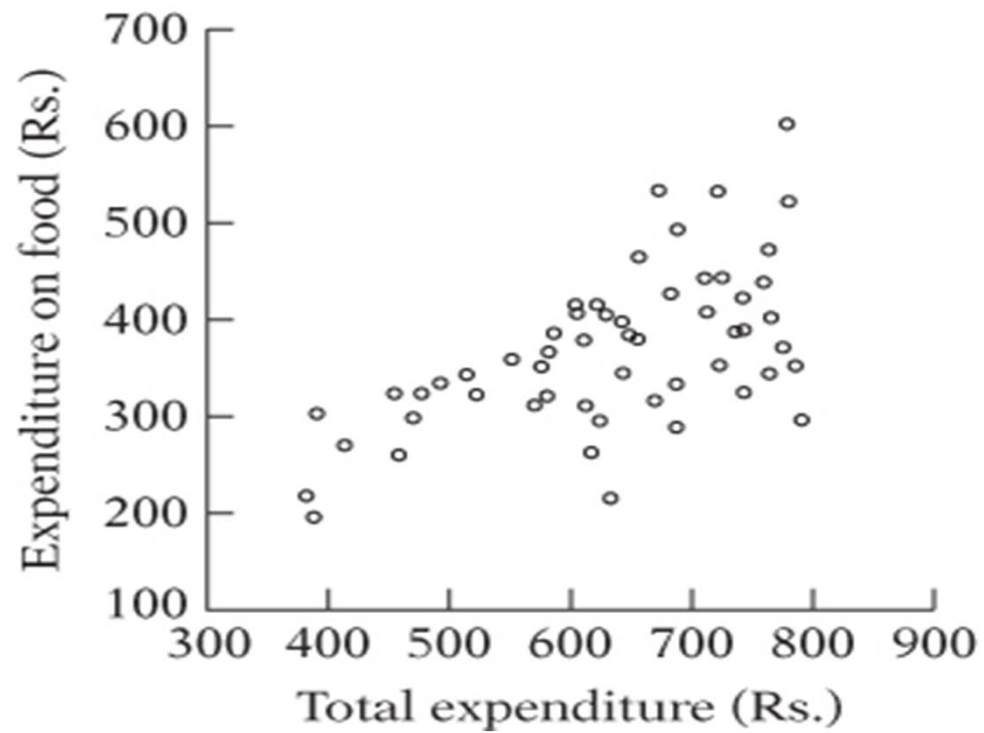


TABLE 2.8 Food and Total Expenditure (Rupees)

Observation	Food Expenditure	Total Expenditure	Observation	Food Expenditure	Total Expenditure
1	217.0000	382.0000	29	390.0000	655.0000
2	196.0000	388.0000	30	385.0000	662.0000
3	303.0000	391.0000	31	470.0000	663.0000
4	270.0000	415.0000	32	322.0000	677.0000
5	325.0000	456.0000	33	540.0000	680.0000
6	260.0000	460.0000	34	433.0000	690.0000
7	300.0000	472.0000	35	295.0000	695.0000
8	325.0000	478.0000	36	340.0000	695.0000
9	336.0000	494.0000	37	500.0000	695.0000
10	345.0000	516.0000	38	450.0000	720.0000
11	325.0000	525.0000	39	415.0000	721.0000
12	362.0000	554.0000	40	540.0000	730.0000
13	315.0000	575.0000	41	360.0000	731.0000
14	355.0000	579.0000	42	450.0000	733.0000
15	325.0000	585.0000	43	395.0000	745.0000
16	370.0000	586.0000	44	430.0000	751.0000
17	390.0000	590.0000	45	332.0000	752.0000
18	420.0000	608.0000	46	397.0000	752.0000
19	410.0000	610.0000	47	446.0000	769.0000
20	383.0000	616.0000	48	480.0000	773.0000
21	315.0000	618.0000	49	352.0000	773.0000
22	267.0000	623.0000	50	410.0000	775.0000
23	420.0000	627.0000	51	380.0000	785.0000
24	300.0000	630.0000	52	610.0000	788.0000
25	410.0000	635.0000	53	530.0000	790.0000
26	220.0000	640.0000	54	360.0000	795.0000
27	403.0000	648.0000	55	305.0000	801.0000
28	350.0000	650.0000			

Source: Chandan Mukherjee, Howard White, and Marc Wuyts, *Econometrics and Data Analysis for Developing Countries*, Routledge, New York, 1998, p. 457.

$$\widehat{Foodexp}_i = -1,283.912 + 257.2700 \ln Totalexp_i$$

- ▶ As this figure suggests, food expenditure increases more slowly as total expenditure increases, perhaps giving credence to Engel's law
- ▶ The slope coefficient of about 257 means that an increase in the total food expenditure of 1 percent, on average, leads to about 2.57 rupees increase in the expenditure on food of the 55 families included in the sample

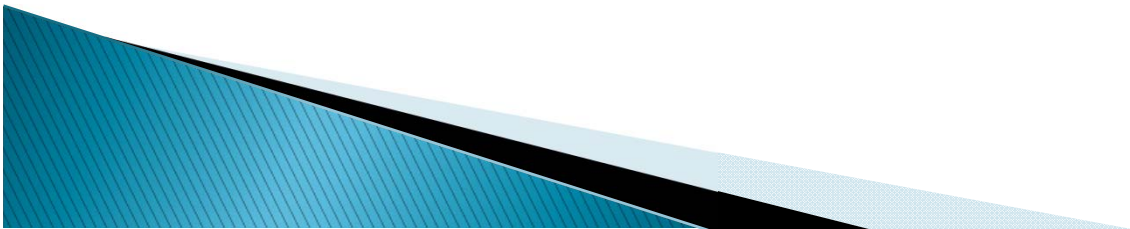
(Note: we have divided the estimated slope coefficient by 100)



Reciprocal models

$$Y_i = \beta_1 + \beta_2 \left(\frac{1}{X_i} \right) + u_i$$

As X increases indefinitely, in term $\beta_2 \left(\frac{1}{X_i} \right)$ approaches zero and Y approaches the limiting or asymptotic value β_1



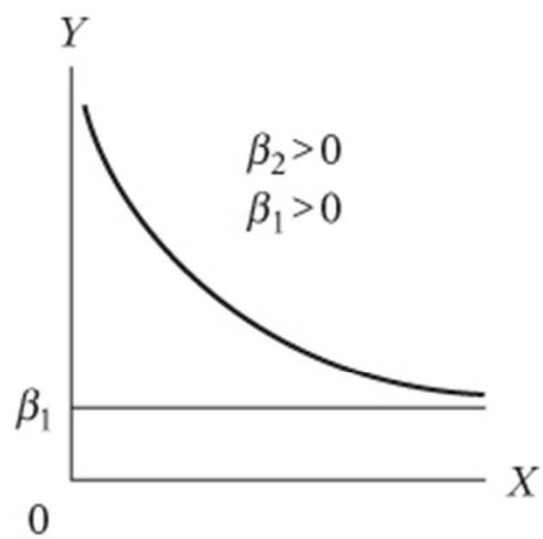
$$Y_i = \beta_1 + \beta_2 \left(\frac{1}{X} \right)$$

$$\frac{dY}{dX} = -\beta_2 \left(\frac{1}{X^2} \right)$$

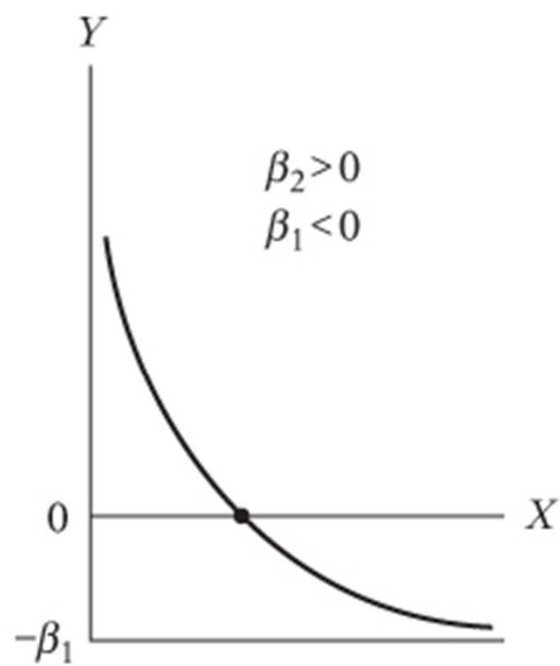
if β_2 is positive, the slope is negative throughout

β_2 is negative, the slope is positive throughout

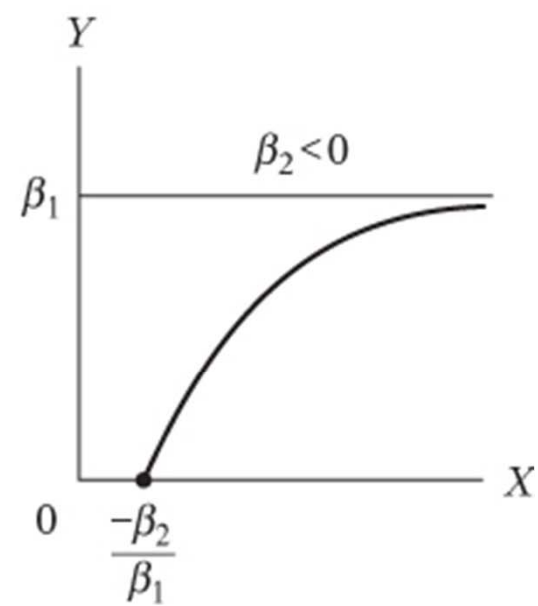




(a)



(b)



(c)

Example

- ▶ Child mortality and per capita GNP – 64 countries

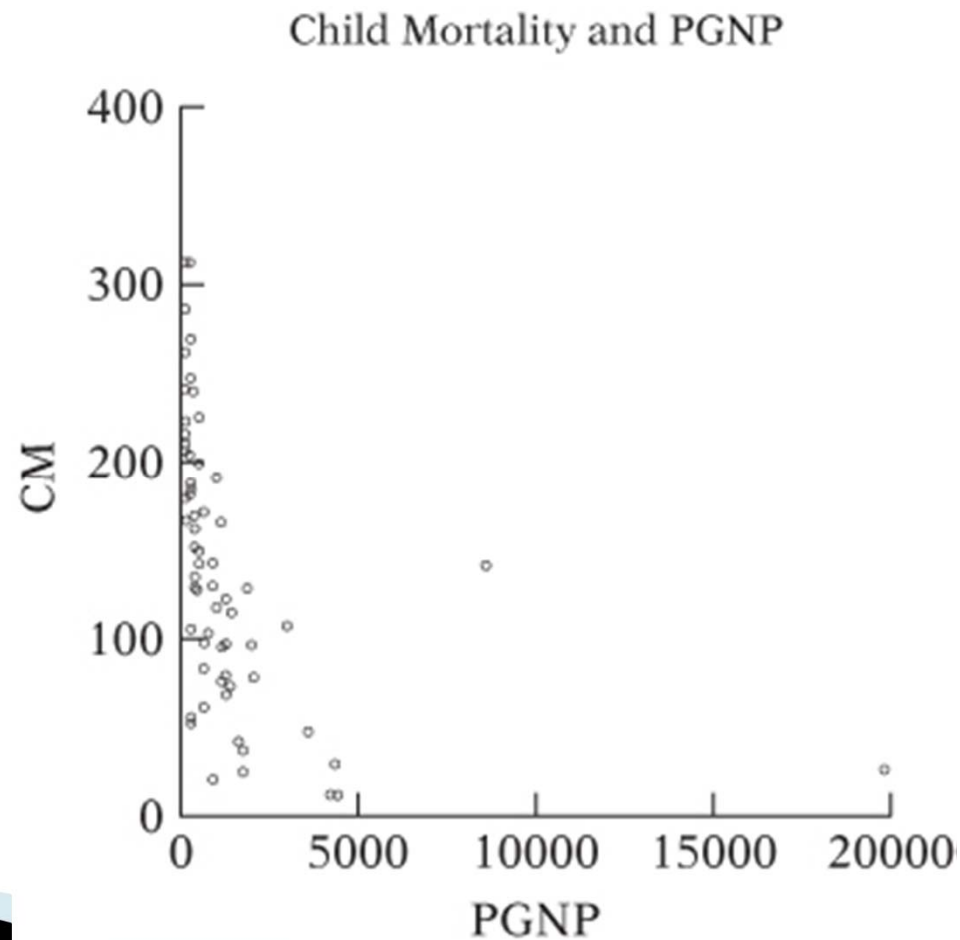


TABLE 6.4 Fertility and Other Data for 64 Countries

Observation	CM	FLFP	PGNP	TFR	Observation	CM	FLFP	PGNP	TFR
1	128	37	1870	6.66	33	142	50	8640	7.17
2	204	22	130	6.15	34	104	62	350	6.60
3	202	16	310	7.00	35	287	31	230	7.00
4	197	65	570	6.25	36	41	66	1620	3.91
5	96	76	2050	3.81	37	312	11	190	6.70
6	209	26	200	6.44	38	77	88	2090	4.20
7	170	45	670	6.19	39	142	22	900	5.43
8	240	29	300	5.89	40	262	22	230	6.50
9	241	11	120	5.89	41	215	12	140	6.25
10	55	55	290	2.36	42	246	9	330	7.10
11	75	87	1180	3.93	43	191	31	1010	7.10
12	129	55	900	5.99	44	182	19	300	7.00
13	24	93	1730	3.50	45	37	88	1730	3.46
14	165	31	1150	7.41	46	103	35	780	5.66
15	94	77	1160	4.21	47	67	85	1300	4.82
16	96	80	1270	5.00	48	143	78	930	5.00
17	148	30	580	5.27	49	83	85	690	4.74
18	98	69	660	5.21	50	223	33	200	8.49
19	161	43	420	6.50	51	240	19	450	6.50
20	118	47	1080	6.12	52	312	21	280	6.50
21	269	17	290	6.19	53	12	79	4430	1.69
22	189	35	270	5.05	54	52	83	270	3.25
23	126	58	560	6.16	55	79	43	1340	7.17
24	12	81	4240	1.80	56	61	88	670	3.52
25	167	29	240	4.75	57	168	28	410	6.09
26	135	65	430	4.10	58	28	95	4370	2.86
27	107	87	3020	6.66	59	121	41	1310	4.88
28	72	63	1420	7.28	60	115	62	1470	3.89
29	128	49	420	8.12	61	186	45	300	6.90
30	27	63	19830	5.23	62	47	85	3630	4.10
31	152	84	420	5.79	63	178	45	220	6.09
32	224	23	530	6.50	64	142	67	560	7.20

Note: CM = Child mortality, the number of deaths of children under age 5 in a year per 1000 live births.

FLFP = Female literacy rate, percent.

PGNP = per capita GNP in 1980.

TFR = total fertility rate, 1980–1985, the average number of children born to a woman, using age-specific fertility rates for a given year.

Source: Chandan Mukherjee, Howard White, and Marc Whyte, *Econometrics and Data Analysis for Developing Countries*, Routledge, London, 1998, p. 456.

$$\widehat{CM}_i = 81.7944 + 27,237.17 \left(\frac{1}{PGNP_i} \right)$$

- ▶ As per capita GNP increases, one would expect child mortality to decrease because people can afford to spend more on health care, assuming all other factors remain constant
- ▶ The relationship is not a straight line one: As per capita GNP increases, initially there is a dramatic drop in child mortality but the drop tapers off as per capita GNP continues to increase



As per capita GNP increases indefinitely, child mortality approaches its asymptotic value of about 82 deaths per thousand



Example

The Phillips curve

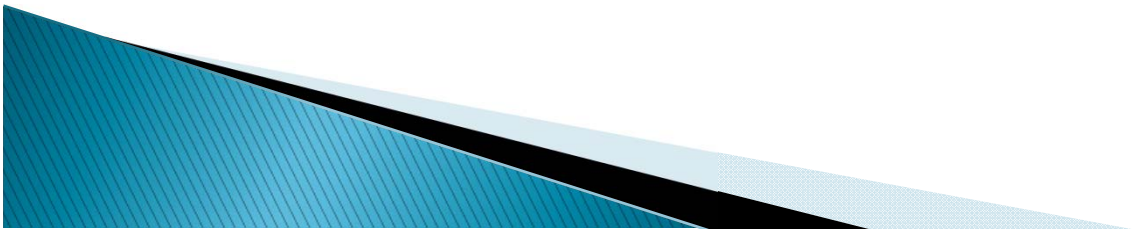
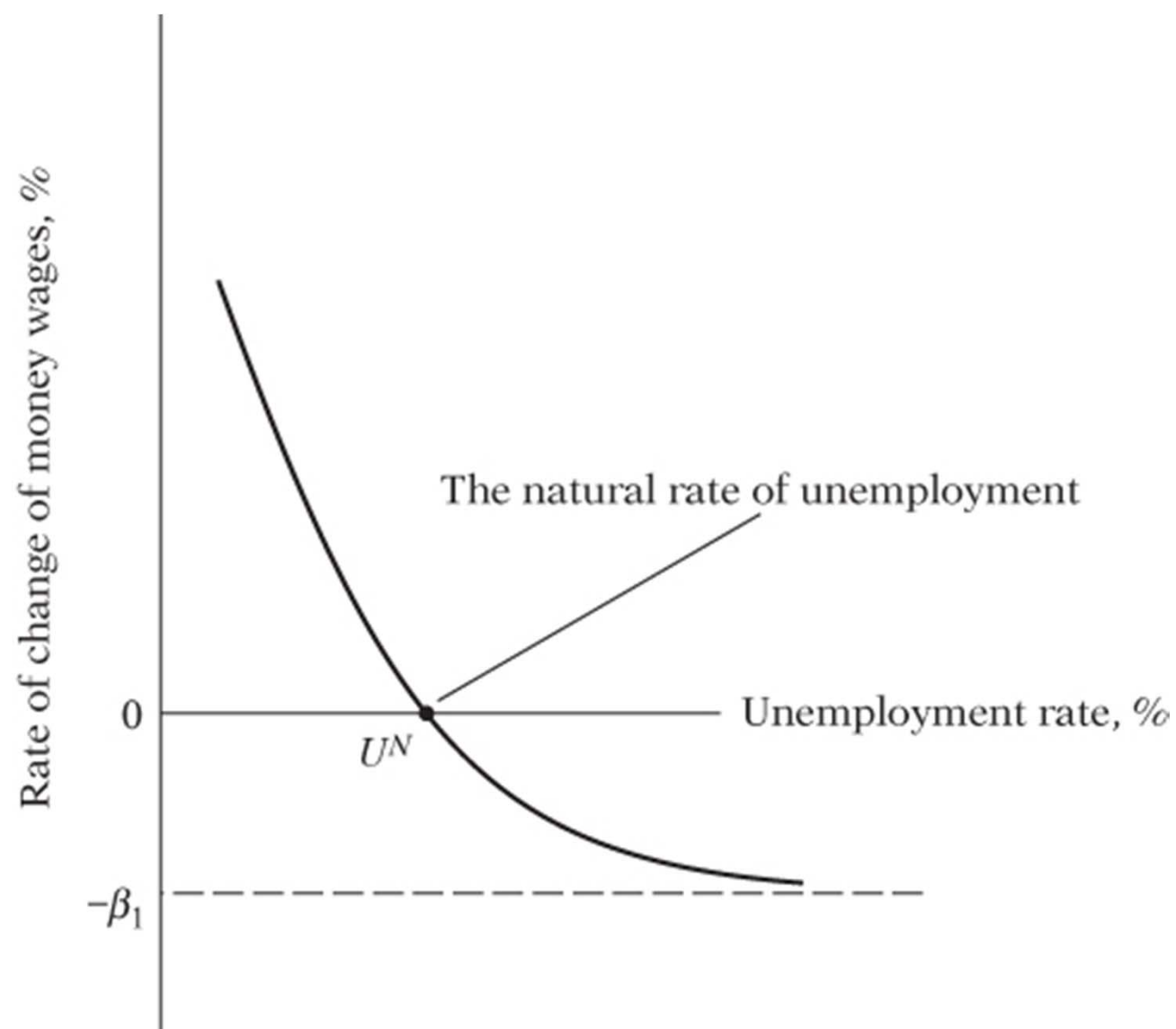


Figure 6.8



- ▶ Using the data on percent rate of change of money wages (Y) and the unemployment rate (X) for the United Kingdom for the period 1861-1957, Phillips obtained a curve whose general shape resembled figure 6.8
- ▶ An asymmetry in the response of wage changes to the level of the unemployment rate: Wages rise faster for a unit change in unemployment if the unemployment rate is below U^N , which is called **the natural rate of unemployment**, and then they fall slowly for an equivalent change when the unemployment rate is above the natural rate, indicating the asymptotic floor, or β_1 for wage change



- ▶ This particular feature of the Phillips curve may be due to institutional factors such as union bargaining power, minimum wages, unemployment compensation.
- ▶ A comparative recent formulation is provided by Olivier Blanchard

$$\pi_t - \pi_t^e = \beta_2(UN_t - U^N) + u_t$$

Where

π_t = actual inflation rate at time t

π_t^e = expected inflation rate at time t, the expectation being formed in year (t-1)

UN_t = actual unemployment rate prevailing at time t

U^N = natural rate of unemployment

u_t = stochastic error term



- ▶ Since π_t^e is not directly observable, as a starting point one can make the simplifying assumption that $\pi_t^e = \pi_{t-1}$; that is, the inflation rate expected this year is the inflation rate that prevailing in the last year
- ▶ Substituting this assumption and writing the regression model in the standard form, we obtain

$$\pi_t - \pi_{t-1} = \beta_1 + \beta_2 UN_t + u_t$$

$$\text{where } \beta_1 = -\beta_2 U^N$$

Equation states that the change in the inflation rate between two time periods is linearly related to the current unemployment rate



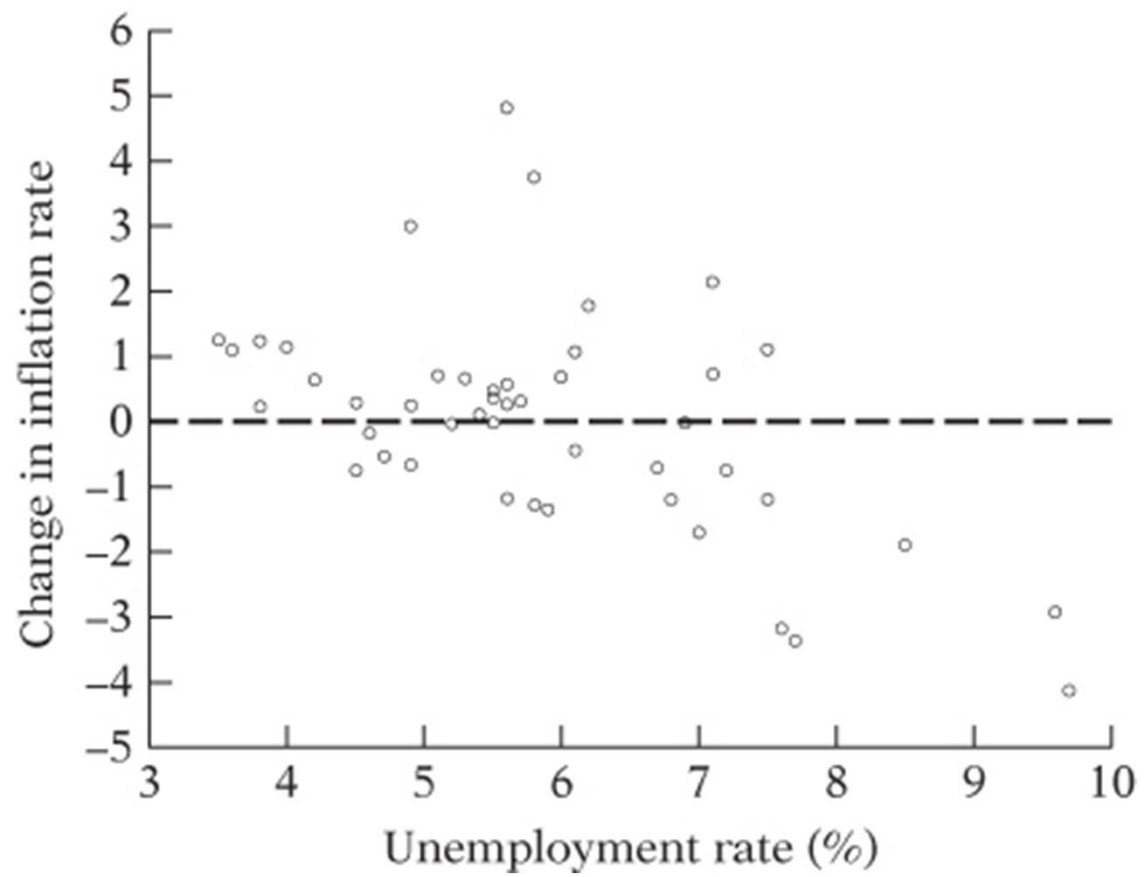
TABLE 6.5

**Inflation Rate and
Unemployment
Rate, United States,
1960–2006
(For all urban
consumers;
1982–1984 = 100,
except as noted)**

Source: *Economic Report of
the President*, 2007, Table
B-60, p. 399, for CPI changes
and Table B-42, p. 376, for
the unemployment rate.

Year	INFLRATE	UNRATE	Year	INFLRATE	UNRATE
1960	1.718	5.5	1984	4.317	7.5
1961	1.014	6.7	1985	3.561	7.2
1962	1.003	5.5	1986	1.859	7.0
1963	1.325	5.7	1987	3.650	6.2
1964	1.307	5.2	1988	4.137	5.5
1965	1.613	4.5	1989	4.818	5.3
1966	2.857	3.8	1990	5.403	5.6
1967	3.086	3.8	1991	4.208	6.8
1968	4.192	3.6	1992	3.010	7.5
1969	5.460	3.5	1993	2.994	6.9
1970	5.722	4.9	1994	2.561	6.1
1971	4.381	5.9	1995	2.834	5.6
1972	3.210	5.6	1996	2.953	5.4
1973	6.220	4.9	1997	2.294	4.9
1974	11.036	5.6	1998	1.558	4.5
1975	9.128	8.5	1999	2.209	4.2
1976	5.762	7.7	2000	3.361	4.0
1977	6.503	7.1	2001	2.846	4.7
1978	7.591	6.1	2002	1.581	5.8
1979	11.350	5.8	2003	2.279	6.0
1980	13.499	7.1	2004	2.663	5.5
1981	10.316	7.6	2005	3.388	5.1
1982	6.161	9.7	2006	3.226	4.6
1983	3.212	9.6			

Note: The inflation rate is the percent year-to-year change in CPI. The unemployment rate is the civilian unemployment rate.



- ▶ As expected, the relation between the change in inflation rate and the unemployment rate is negative – a low unemployment rate leads to an increase in the inflation rate and therefore an acceleration of the price level, hence the name accelerationist Phillips curve



Model 1

Linear model:

$$\begin{aligned}(\widehat{\pi_t - \pi_{t-1}}) &= 3.7844 - 0.6385UN_t \\ t &= (4.1912) \quad (-4.2756) \\ r^2 &= 0.2935\end{aligned}$$

Model 2

Reciprocal model:

$$\begin{aligned}(\widehat{\pi_t - \pi_{t-1}}) &= -3.0684 + 17.2077\left(\frac{1}{UN_t}\right) \\ t &= (-3.1635) \quad (3.2886) \\ r^2 &= 0.1973\end{aligned}$$

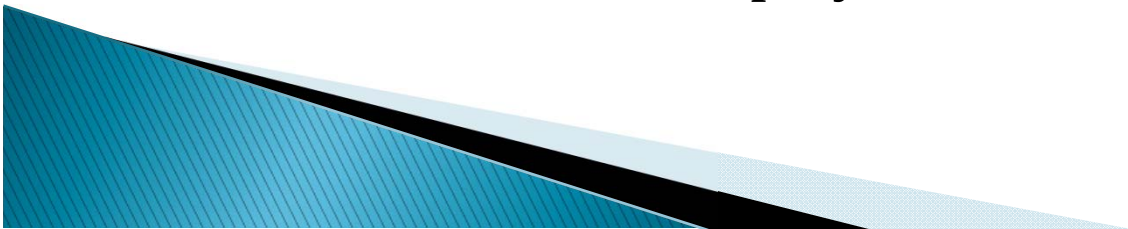


Model 1 shows that if the unemployment rate goes down by 1 percentage point, on average, the change in the inflation rate goes up by about 0.64 percentage points

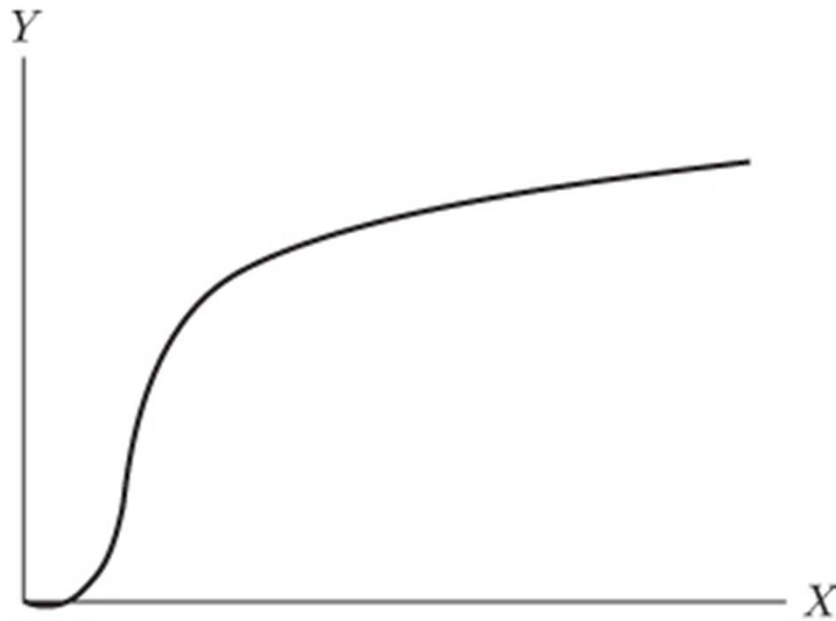
Model 2 shows that even if the unemployment rate increases indefinitely, the most the change in the inflation rate will go down will be about 3.07 percentage points We can compute the underlying natural rate of unemployment as:

$$UN_t = \frac{\widehat{\beta}_1}{-\widehat{\beta}_2} = \frac{3.7844}{0.6385} = 5.9270$$

The natural rate of unemployment is about 5.93%



The logarithmic reciprocal model



$$\ln Y_i = \beta_1 - \beta_2 \left(\frac{1}{X_i} \right) + u_i$$



The logarithmic reciprocal model

- ▶ Initially Y increases at an increasing rate and then it increases at a decreasing rate

- ▶ E.g. short run production function

Microeconomics – if labor and capital are the inputs in a production function and if we keep the capital input constant but increase the labor input, the short-run output-labor relationship will resemble in the logarithmic reciprocal model



Choice of Functional Form

TABLE 6.6

Model	Equation	Slope $\left(= \frac{dY}{dX}\right)$	Elasticity $\left(= \frac{dY}{dX} \frac{X}{Y}\right)$
Linear	$Y = \beta_1 + \beta_2 X$	β_2	$\beta_2 \left(\frac{X}{Y}\right)^*$
Log-linear	$\ln Y = \beta_1 + \beta_2 \ln X$	$\beta_2 \left(\frac{Y}{X}\right)$	β_2
Log-lin	$\ln Y = \beta_1 + \beta_2 X$	$\beta_2 (Y)$	$\beta_2 (X)^*$
Lin-log	$Y = \beta_1 + \beta_2 \ln X$	$\beta_2 \left(\frac{1}{X}\right)$	$\beta_2 \left(\frac{1}{Y}\right)^*$
Reciprocal	$Y = \beta_1 + \beta_2 \left(\frac{1}{X}\right)$	$-\beta_2 \left(\frac{1}{X^2}\right)$	$-\beta_2 \left(\frac{1}{XY}\right)^*$
Log reciprocal	$\ln Y = \beta_1 - \beta_2 \left(\frac{1}{X}\right)$	$\beta_2 \left(\frac{Y}{X^2}\right)$	$\beta_2 \left(\frac{1}{X}\right)^*$

Note: * indicates that the elasticity is variable, depending on the value taken by X or Y or both. When no X and Y values are specified, in practice, very often these elasticities are measured at the mean values of these variables, namely, $\bar{X}$ and $\bar{Y}$.



Source

Gujarati, D.N. (2009) Basic Econometrics. 5th ed. Singapore, McGraw-Hill.

