Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

$$2, 3, 4, 6$$

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E\left[m X_a\right] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

$$P_s = \pi_s m_s 1$$

$$m_s = \frac{\partial u'(C_s^*)}{u'(C_0^*)} = \frac{\partial C_0^*}{C_s^*}$$

$$\therefore \quad P_s = \pi_s \partial \frac{C_0^*}{C_s^*}$$

3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E\left[m_{01} R_s\right]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\widetilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U\left(C\right) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

$$M_{01} = \frac{\partial u'(C_1)}{u'(C_0)} = \partial \left(\frac{C_0}{C_1}\right)^{r-1}$$

$$1 = \pi_s \, \partial \left(\frac{C_0}{C_1}\right)^{r-1} R_s$$

F.O.C. $\quad 0 = \pi_s \, \partial (r-1) \left(\frac{C_0}{C_1}\right)^{r-2} R_s + \pi_s \, \partial \left(\frac{C_0}{C_1}\right)^{r-1} d R_s$

$$\frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{d R_s} = \frac{1}{1-r}$$

4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

$$p = \begin{bmatrix} 1/1.05 \\ 6 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix} \qquad X^{-1} = \frac{1}{-5}\begin{bmatrix} 5 & -10 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1/5 & -1/5 \end{bmatrix}$$

$$[p_1 \; p_2] = p'X^{-1} = [1/1.05 \quad 6]\begin{bmatrix} -1 & 2 \\ 1/5 & -1/5 \end{bmatrix} = [0.2476 \quad 0.7048]$$

risk-neutral probabilities : $p_1 R_f = 0.2476(1.05) = 0.26$
$p_2 R_f = 0.7048(1.05) = 0.74$

b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

$$m_1 = \frac{p_1}{\pi_1} = \frac{0.2476}{0.5} = 0.495$$

$$m_2 = \frac{p_2}{\pi_2} = \frac{0.7048}{0.5} = 1.410$$

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

$$0 = E[m(R_i - R_f)]$$

$$0 = E[(a + bR_m)(R_i - R_f)]$$

$$0 = aE[R_i] + aR_f + bE[R_m R_i] - bR_f E[R_m]$$

$$0 = a[E(R_i) + R_f] + b[E(R_m R_i) + cov(R_m, R_i) - R_f E[R_m]]$$

$$0 = (E[R_i] + R_f)(a + bE(R_m)) + b\,cov(R_m, R_i)$$

$$-b\,cov(R_m, R_i) = (E[R_i] + R_f)(a + bE(R_m))$$

$$E[R_i] + R_f = \frac{-b\,cov(R_m, R_i)}{a + bE(R_m)}$$

$$= \frac{-cov(R_m, R_i)}{\sigma_m^2} \cdot \frac{b\sigma_m^2}{a + bE(R_m)}$$

$$= -\beta_i \frac{b\sigma_m^2}{a + bE(R_m)}$$

$$\therefore \gamma = -\frac{b\sigma_m^2}{a + bE[R_m]}$$

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

$$\frac{1}{R_f} = E[a + bR_m]$$

$$a = \frac{1}{R_f} - bE(R_m)$$

Market port ; $1 = E[(a + bR_m)R_m] \cdot aE(R_m) + bE(R_m^2)$

$$1 = aE(R_m) + bE(\sigma_m^2 + E(R_m)^2)$$

$$1 = \left[\frac{1}{R_f} - bE[R_m]\right][E(R_m) + b(\sigma_m^2 + E(R_m)^2)]$$

$$0 = \frac{E(R_m)}{R_f} + b\sigma_m^2$$

$$b = \frac{-E(R_m) - R_f}{R_f\sigma_m^2}$$

$$a = \frac{\sigma_m^2 + E(R_m)(E(R_m) - R_f)}{R_f\sigma_m^2}$$

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

$$a + bE(R_m) = \frac{\sigma_m^2 + E(R_m)(E(R_m) - R_f) - E(R_m) - R_f}{R_f\sigma_m^2}$$

$$a + bE(R_m) = \frac{1}{R_f}$$

$$R = -\frac{b\sigma_m^2}{a + bE(R_m)}$$

$$= \frac{E(R_m) - R_f}{R_f\sigma_m^2} \cdot R_f\sigma_m^2$$

$$= E(R_m) - R_f$$