EE320 (2/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION WITHOUT CONSTRAINTS:

ONE-INDEPENDENT-VARIABLE CASE

Topics

- Optimal Values and Extreme Values (Maxima, Minima, inflection points)
  - First-derivative test for relative extremum
  - Second-derivative Test for relative extremum
- Convexity and Concavity
- Profit maximization
  - Competitive market
  - Monopoly
- Effects of taxes
  - Lump-sum tax
  - Profit tax
  - Excise
- Tax Revenue Maximization

OPTIMAL VALUES AND EXTREME VALUES

Optimal Values and Extreme Values

- Most common criterion of choice among alternatives in economics is to find an *extreme* value – *maximum* or *minimum*.
- *Optimization* problems could be *maximizing* something (e.g. maximizing profit) or *minimizing* something (e.g. minimizing costs).
  - Objective function → dependent variable
  - Choice → independent variable
- Example: Firm's objective is to maximize profit:

$$\pi(Q) = R(Q) - C(Q).$$

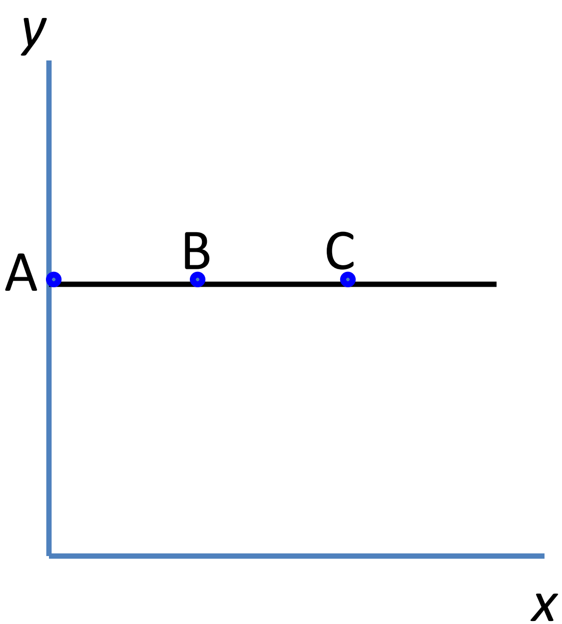
→ π is the object of maximization

→ Q is the choice variable.

- This lecture will focus on the general objective function of one variable: $y = f(x)$.

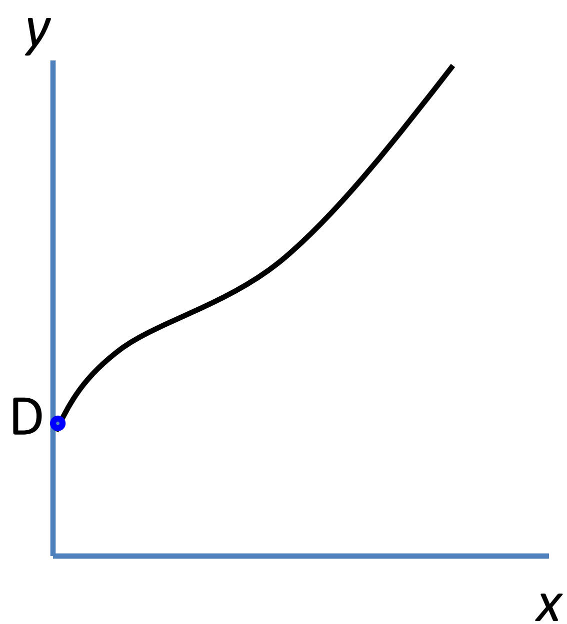
Relative VS. Absolute Extremum

(a)



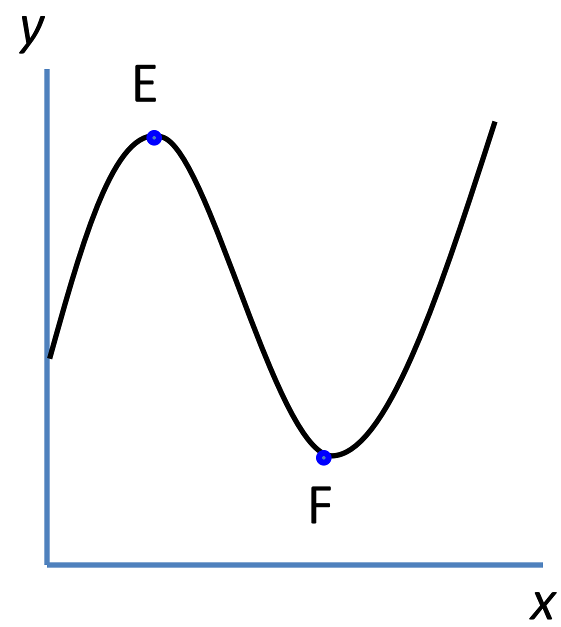
No extremum

(b)



Absolute (or global) minimum

(c)



Relative (or local) extremums

First-Derivative Test (1)

- 2 cases if a relative extremum of $f(x)$ occurs at $f(x_0)$:
  1. $f'(x_0)$ does not exist.
  2. $f'(x_0) = 0$.

Figure 1

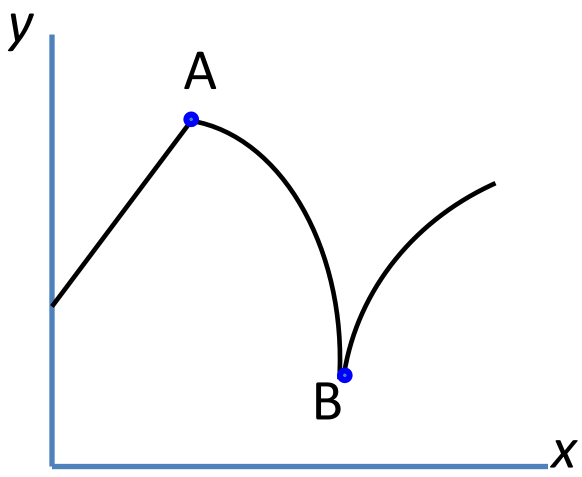
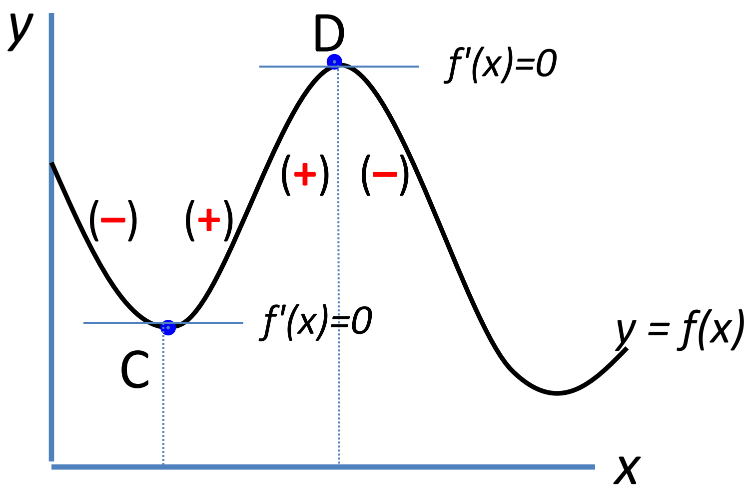


Figure 2



For smooth functions, the *necessary* condition to determine relative extreme values (local max/local min) is $f'(x) = 0$.

First-Derivative Test (2)

First-Derivative Test for Relative Extremum:

If the first derivative of a function f(x) at $x = x_0$ is $f'(x_0) = 0$, then the value of $f(x_0)$ will be

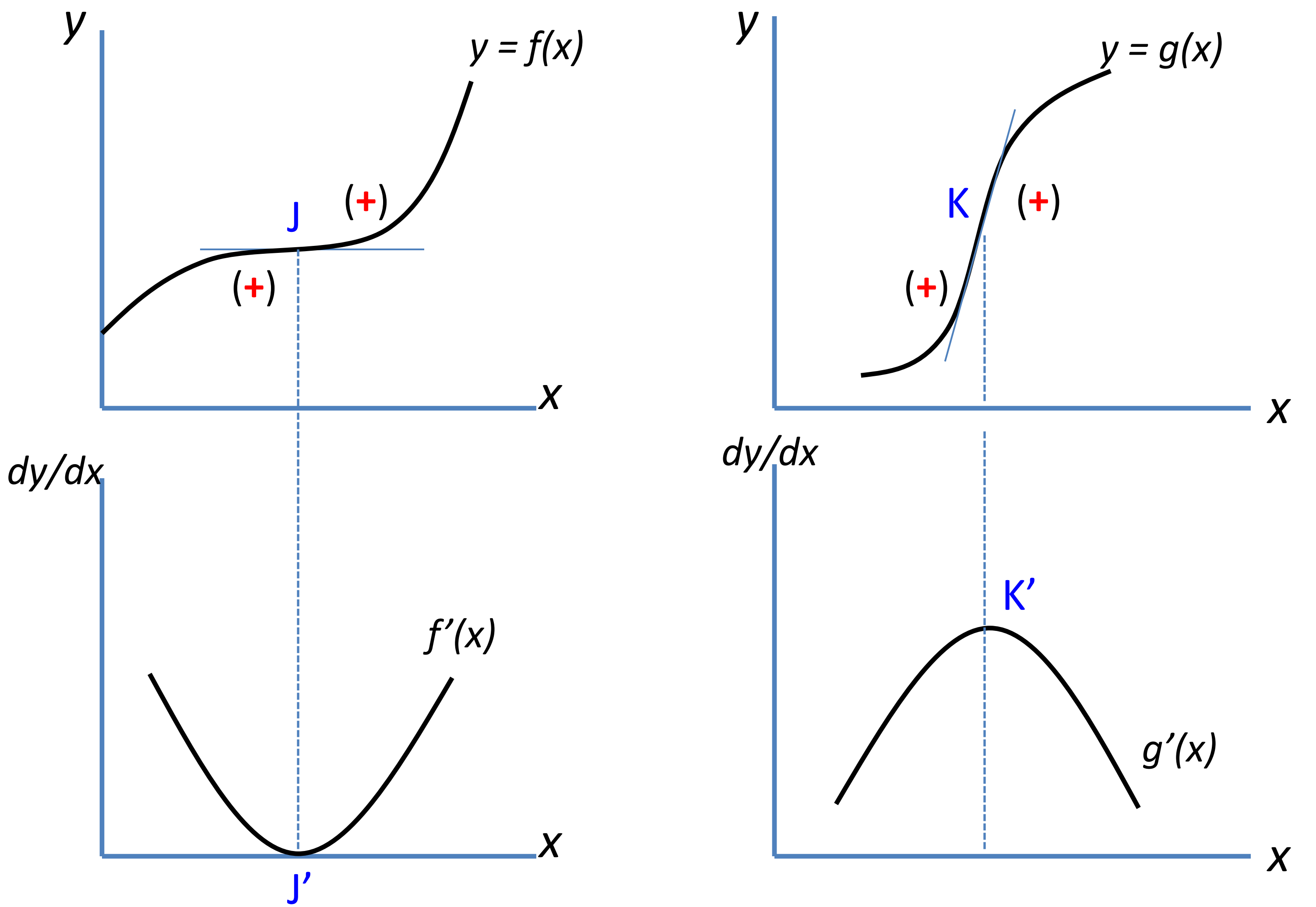
a. A relative *maximum* if $f'(x)$ changes its sign from + to – from the immediate left of x_0 to its immediate right.

b. A relative *minimum* if $f'(x)$ changes its sign from – to + from the immediate left of x_0 to its immediate right.

c. *Neither* a relative maximum nor a relative minimum if f'(x) has the same sign on both the immediate left and the immediate right of x_0. → *x_0 is an inflection point.*

→ x_0 where $f'(x_0) = 0$ is called the critical point, and $f(x_0)$ is the stationary value of the function f(x).

Inflection Points



Example

- Find the relative minimum of the average cost function

$$AC(Q) = Q^2 – 5Q + 8$$

➔ Let $f(Q) = AC(Q)$

$f'(Q) = 2Q – 5$

Necessary condition for an extremum: $2Q – 5 = 0$

➔ $Q^* = 2.5$

Check: if $Q = 2$, $f'(x) = -1$,

if $Q = 3$, $f'(x) = 1$.

Thus, $Q^* = 2.5$ is a minimum.

Second and Higher Derivatives

- The derivative of a function *f* is called the first derivative of *f*.
- If *f′* is also differentiable, we can differentiate *f′* to get the second derivative of *f*:
- Definition:

$$f''(x) \equiv \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

- Alternative notations: $\frac{d^2 f(x)}{dx^2}, \frac{d^2 y}{dx^2}, y''$
- As long as the differentiability condition is met, higher-order derivatives can be written similarly as:

$$f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x) \quad \text{or} \quad \frac{d^3 y}{dx^3}, \frac{d^4 y}{dx^4}, \ldots, \frac{d^n y}{dx^n}$$

Interpretation of the Second Derivative

- First derivative:

  $f'(x_0) > 0$: the *value of the function* tends to *increase*.

  $f'(x_0) < 0$: the *value of the function* tends to *decrease*.

- Second derivative:

  $f''(x_0) > 0$: the *slope of the curve* tends to *increase*.

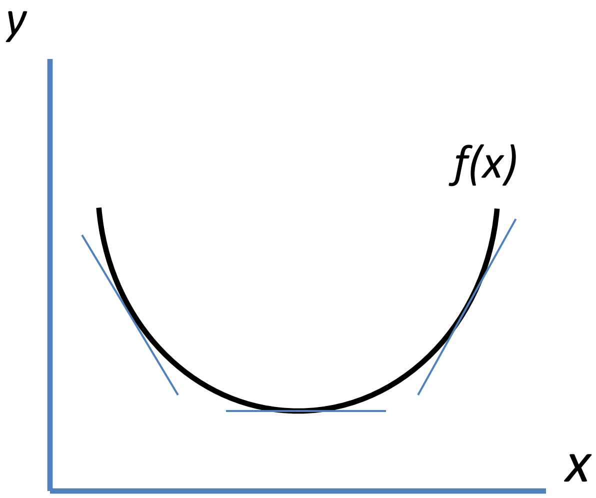
  $f''(x_0) < 0$: the *slope of the curve* tends to *decrease*.

- Possible cases:
  - $f'(x) > 0$ and $f''(x) > 0$: the slope is *positive and increasing* as x increases.
  - $f'(x) < 0$ and $f''(x) > 0$: the slope is *negative and increasing* as x increases.
  - $f'(x_0) > 0$ and $f''(x) < 0$: the slope is *positive and decreasing* as x increases.
  - $f'(x_0) < 0$ and $f''(x) < 0$: the slope is *negative and decreasing* as x increases.

Curvature of a Graph

$f''(x) > 0$:

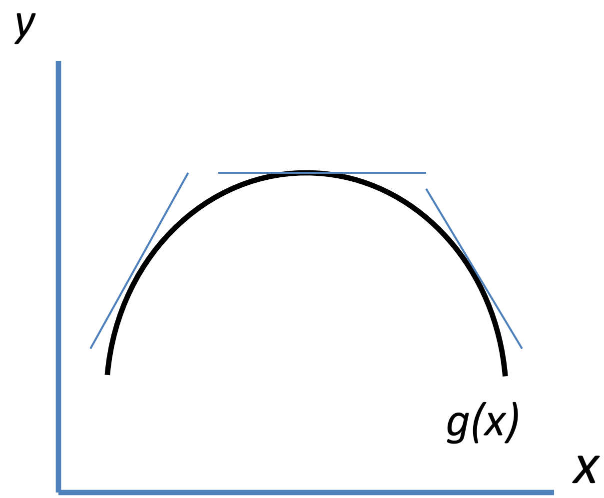
Slope of the tangent increases as x increases (i.e. $f'(x)$ is increasing).



⇨ Convex Function

$g''(x) < 0$:

Slope of the tangent decreases as x increases (i.e. $g'(x)$ is decreasing).



⇨ Concave Function

Convexity and Concavity (1)

Definition: *Strictly Convex*

A function *f(x)* is *strictly convex* if we pick any pair of points *M* and *N* on its curve and join them by a straight line, the line segment *MN* must lie entirely *above* the curve, excepts at points *M* and *N*.

Definition: *Strictly Concave*

A function *f(x)* is *strictly concave* if we pick any pair of points *M* and *N* on its curve and join them by a straight line, the line segment *MN* must lie entirely *below* the curve, excepts at points *M* and *N*.

Note: When the segment MN *either* lies above (below) *or* lies along the curve, the function is *convex (concave)*.

Convexity and Concavity (2)

- Definition: Assume that f is continuous and twice differentiable.
  If $f''(x) > 0$ for all x, then $f(x)$ is a *strictly convex* function.
  If $f''(x) < 0$ for all x, then $f(x)$ is a *strictly concave* function.

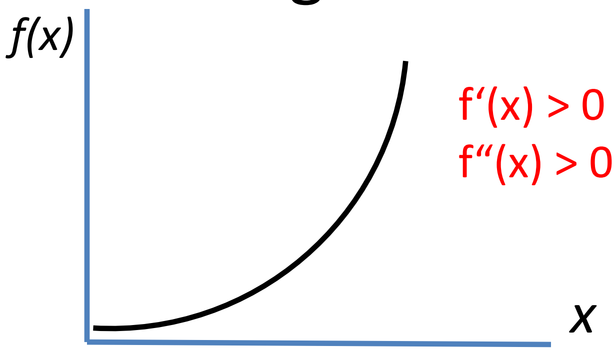
- Example 1: $f(x) = x^2 - 2x + 2$
  - $f'(x) = 2x - 2 \Rightarrow f''(x) = 2 > 0$.
  - Thus, f(x) is a strictly convex function.

- Example 2: $f(x) = ax^2 + bx + c$
  - $f'(x) = 2ax + b \Rightarrow f''(x) = 2a$.
  - Thus, f(x) is strictly convex if $a > 0$ and f(x) is strictly concave if $a < 0$.

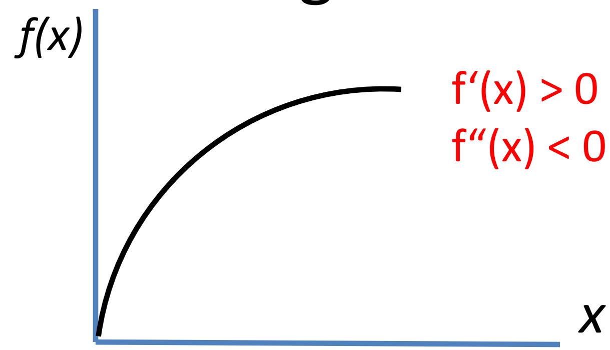
Note: In certain cases, $f''(x)$ may have a zero value at the stationary point. Thus, the second derivative is a sufficient but not necessary condition. (Example: $f(x) = x^4$).

Convexity and Concavity (3)

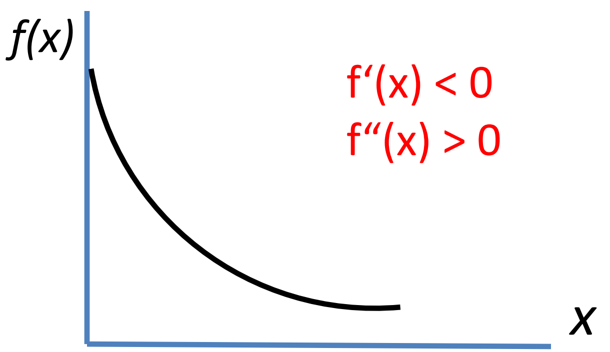
- Increasing Convex



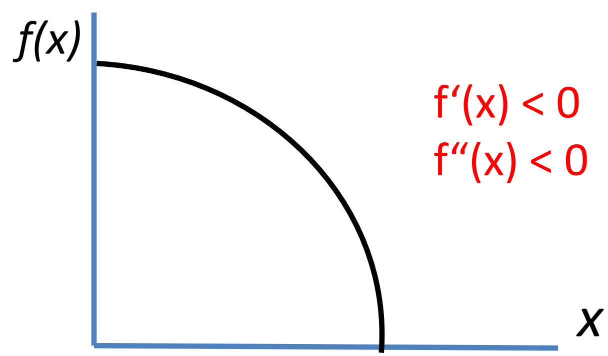
- Increasing Concave



- Decreasing Convex

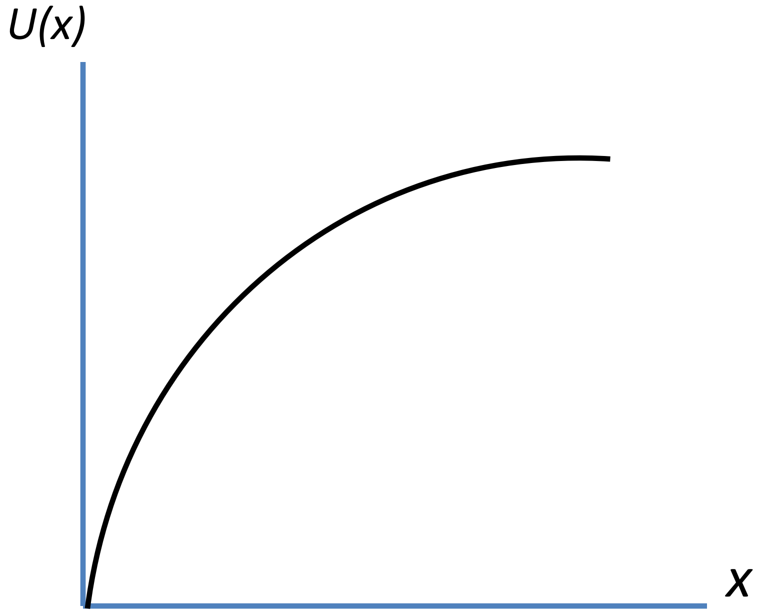


- Decreasing Concave

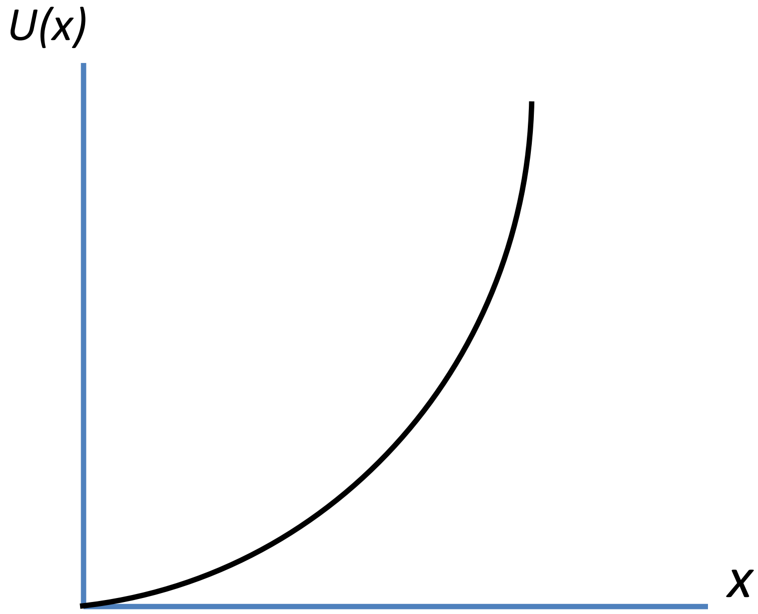


Example: Attitudes toward Risk

Risk-Averse Preference



Risk-Loving Preference



Second-Derivative Test

Second-Derivative Test for Relative Extremum:

If $f'(x_0) = 0$, then the value of $f(x_0)$ will be

a. A relative *maximum* if $f''(x) < 0$.

b. A relative *minimum* if $f''(x) > 0$.

Example 1: Determine the extremum of $f(x) = x^3 - 3x^2 + 2$

- $f'(x) = 3x^2 - 6x = 0$ → $x(x-2) = 0$ → $x^* = 0, 2$.
- $f''(x) = 6x - 6$ → $f''(0) = -6$ → $f(0)$ is max ; $f''(2) = 6 > 0$ → $f(2)$ is min

Example 2: Determine the extremum of $g(x) = \frac{x^4}{4} - \frac{3}{2}x^2$

- $f'(x) = x^3 - 3x = 0$ → $x(x^3-3) = 0$ → $x^* = 0, \pm\sqrt{3}$
- $f''(x) = 3x^2 - 3$ → $f''(0) = -3$ → $f(0)$ is max ; $f''(3^{1/2}) = f''(-3^{1/2}) = 6 > 0$ → $f(3^{1/2})$ and $f(-3^{1/2})$ are min.

Necessary VS. Sufficient Conditions for Relative Extremum

Condition	Maximum	Minimum
First-order necessary	$f'(x) = 0$	$f'(x) = 0$
Second-order necessary	$f''(x) \leq 0$	$f''(x) \geq 0$
Second-order sufficient	$f''(x) < 0$	$f''(x) > 0$

Example: Determine the extremum of $f(x) = x^4$

- *FOC:* $f'(x) = 4x^3 = 0$ when $x^* = 0$.
- *SOC:* $f''(x) = 12x^2 \geq 0$ is a second-order necessary condition for a minimum.

OPTIMIZATION PROBLEMS

Optimization without Constraints

- Main application of finding derivatives is to solve optimization problems.
- The essence of the optimization problem is to choose the best alternative available.
- Two types of optimization problems:
  - Optimization <u>without</u> constraints**
    - Profit maximization
    - Tax revenue maximization
  - Optimization <u>with</u> constraints (after midterm)

Profit Maximization

- Profit function: $\pi(Q) = R(Q) - C(Q)$
- First-order necessary condition:

  $\pi'(Q) = R'(Q) - C'(Q) = 0$

  ⇔ MR (Q) = MC(Q)

  (i.e. slope of revenue function = slope of cost function)
- Second-order sufficient condition:

  $\pi''(Q^*) < 0$

  ⇔ $MR'(Q) < MC'(Q)$
- Profit-maximizing output:

  Q* such that $\pi'(Q^*) = 0$.

Profit Maximization: Perfect Competition

- Let the profit function be $\pi = TR - TC$ where $TR=R(Q\ ); TC=C(Q)$
- Objective function: $\max_Q \pi = R(Q) - C(Q)$
- Perfectly competitive market: *P* is constant.

  ➔ Total revenue: $TR = P \times Q$
- First-order necessary condition:

  $$\pi'(Q) = R'(Q) - C'(Q) = 0$$

  $$\Leftrightarrow \quad P = MC(Q)$$
- Second-order sufficient condition:

  $$R''(Q) = 0 \ \ \& \ C''(0) > 0$$

  $$\Leftrightarrow \pi''(Q^*) < 0$$

Graph: π-max in perfect competition

Example: π-max in perfect competition

- Let P = 30; TC = 100 + 19Q – 5Q^2 + (1/3)Q^3. Find Q* which maximizes profit.

Profit Maximization: Monopoly

- Monopoly: Producer/seller can set the price in the market.
- Let $P = f(Q) = a - bQ$
  - Total revenue:

    $$TR = P \times Q = [a-bQ]Q = aQ - bQ^2$$

  - Marginal revenue:

    $$MR = d(TR)/dQ = a - 2bQ$$

- Profit maximization:
  - <u>Objective function</u>:

    $$\max_Q \pi = TR(Q) - TC(Q)$$

Graph: π-max in monopoly market

Example: Profit Maximization

Let $R(Q) = 4000Q - 33Q^2$

$C(Q) = 2Q^3 - 3Q^2 + 400Q + 5000$

Determine the profit maximizing output level.

FONC: $\pi'(Q) = R'(Q) - C'(Q) = 0$

$\Leftrightarrow (4000Q - 33Q^2) - (2Q^3 - 3Q^2 + 400Q + 5000) = 0$

$\Leftrightarrow Q^* = 20$

SOSC: $\pi''(20) = -300 < 0.$

Thus, $Q^* = 20$ is the profit-maximizing output level.

Example: π-max in monopoly market

- Let $P = 48 - 0.5Q$; $TC = 2 + 60Q - 8Q^2 + Q^3$. Find Q^* which maximizes profit.

Effect of Taxes

- Given that $\pi = TR(Q) - TC(Q)$, what will happen to profit-maximizing quantity and price if the government imposes:
  - Lump-sum tax
    - *Fixed* tax amount, e.g. $T = t_0$.
  - Profit tax
    - Tax changes according to profit: $T = t\pi$, where $0 < t < 1$.
  - Specific (or excise) tax
    - Tax varies with quantity: $T = tQ$, where $0 < t < 1$.

Effect of Taxes: Lump-Sum Tax

- Net profit after lump-sum tax: $\pi_N = TR - TC - t_0$
- Objective function:

$$\max_Q \pi_N = TR(Q) - TC(Q) - t_0$$

- Profit-maximizing condition:
- Effect of lump-sum tax on profit-max quantity:

Effect of Taxes: Profit Tax

- Net profit after profit tax: $\pi_N = TR - TC - t\pi = (1\text{-}t)[TR - TC]$
- Objective function:

$$\max_Q \pi_N = (1\text{-}t)[TR(Q) - TC(Q)]$$

- Profit-maximizing condition:
- Effect of profit tax on profit-max quantity:

Effect of Taxes: Excise (or Specific) Tax

- Net profit after tax: $\pi_N = TR - TC - tQ$
- Objective function:

$$\max_Q \pi_N = TR(Q) - TC(Q) - tQ$$

- Profit-maximizing condition:
- Effect of specific tax on profit-max quantity:

Tax Revenue Maximization (1)

- Let $P = a - bQ$, so that $TR = aQ - bQ^2$.
- Let $TC = c_0 + c_1Q + c_2Q^2$.
- Suppose the government imposes a <u>specific tax *t* baht per unit</u>.

➢ After-tax total cost is: $TC_T = c_0 + c_1Q + c_2Q^2 + tQ$

➢ After-tax profit: $\pi_T = aQ - bQ^2 - (c_0 + c_1Q + c_2Q^2 + tQ)$

➢ Profit-maximizing condition:

➢ *Total* tax revenue:

Tax Revenue Maximization (2)

- Tax-revenue-maximizing condition:

- **Example**: Let $P = 40 - 0.5Q$, and $TC = 2 - 5Q + 7Q^2$. Find the specific tax rate *t* that maximizes the government tax revenue.