

Chapter 4 Inequalities without Absolute Values

We are all accustomed to using the sign for equality, [=] as used in the expression $3 \times 4 = 12$ to show that the quantities on the left and right of the sign are equal. However, in many practical situations we need a way of showing, not that two quantities are equal, but that one is bigger or smaller than another. For example, a course may have a limit of 50 students, so that the number of students recruited must be no more than 50; we may have advance orders for 120 of a certain product, so that number produced must be at least 120; and so on.

There are four signs used to represent these inequalities, as they are called:

- $a < b$ means “a is less than b”.
- $a > b$ means “a is greater than b”.
- $a \leq b$ means “a is less than or equal to b”.
- $a \geq b$ means “a is greater than or equal to b”.

In each case the larger end of the sign is directed towards the larger quantity. You are perhaps more likely to encounter the last two signs in practical applications. So we could express the statements in the last paragraph as “number of students ≤ 50 ” and “number of items produced ≥ 120 ”.

Definition 4.1: For any real numbers a and b :

- 1) a is greater than 0 if and only if a is positive real number, denoted by $a > 0$.
- 2) $a > b$ if and only if there exists a real number p such that $a = b + p$ or $a - b = p$.

For example, by Definition 4.1 part (2) we say that $5 > 1$ because $5 = 1 + 4$. In this case the real number p is 4.

Definition 4.2: For any real numbers a and b :

- 1) a is less than 0 if and only if a is negative real number and is denoted by $a < 0$.
- 2) $a < b$ if and only if $b > a$.

Properties of Real Numbers:

1. Closure property under addition: $a + b$ is real
Moreover, if $a > 0$ and $b > 0$, then $a + b > 0$
2. Closure property under multiplication: ab is real
3. Commutative property under addition: $a + b = b + a$
4. Commutative property under multiplication: $ab = ba$
5. Associative property under addition: $(a + b) + c = a + (b + c)$
6. Associative property under multiplication: $(ab)c = a(bc)$
7. Distributive property: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$
8. $a < 0$ if and only if $-a > 0$
9. $a > 0$ if and only if $-a < 0$
10. $(-1)a = -a$
11. $-(-a) = a$
12. $(-a)b = a(-b) = -(ab)$
13. $(-a)(-b) = ab$
14. If $a > 0$ then $\frac{1}{a} > 0$
15. If $a < 0$ then $\frac{1}{a} < 0$
16. $a = b$ or $a < b$ or $a > b$ can be true only for one case.
17. $ab > 0$ if and only if $(a > 0 \text{ and } b > 0)$ or $(a < 0 \text{ and } b < 0)$
18. $ab < 0$ if and only if $(a > 0 \text{ and } b < 0)$ or $(a < 0 \text{ and } b > 0)$

Definition 4.3: For any real numbers a and b :

- 1) a is greater than or equal to b if and only if $a > b$ or $a = b$, denoted by $a \geq b$.
- 2) a is less than or equal to b if and only if $a < b$ or $a = b$, denoted by $a \leq b$.

Definition 4.4: For any real numbers a and b :

- 1) a is not greater than b means $a \leq b$, denoted by $a \nless b$.
- 2) a is not less than b means $a \geq b$, denoted by $a \nless b$.

Section 4.1: Theorems about Inequalities without Absolute Values

There are several important theorems which we can use to solve the inequalities without absolute values.

Theorem 4.1.1: For any real numbers a and b :

- 1) $a \leq a$
- 2) $a \leq b$ and $b \leq a$ if and only if $a = b$

Theorem 4.1.2: For any real numbers a, b, c , and d :

- 1) If $a < b$ and $b < c$, then $a < c$.
- 2) If $a < b$ then $a + c < b + c$.
- 3) If $a < b$ and $c < d$, then $a + c < b + d$.
- 4) If $a < b$ and $c > 0$, then $ac < bc$.
- 5) If $a < b$ and $c < 0$, then $ac > bc$.

Theorem 4.1.3: For any real numbers a, b, c , and d :

- 1) If $a \leq b$ and $b \leq c$, then $a \leq c$.
- 2) If $a \leq b$ then $a + c \leq b + c$.
- 3) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.
- 4) If $a \leq b$ and $c > 0$, then $ac \leq bc$.
- 5) If $a \leq b$ and $c < 0$, then $ac \geq bc$.

Theorem 4.1.4: For any real numbers a, b, c , and d where $0 < a < b$, $0 < c < d$, and n is any positive integer:

- 1) $0 < ac < bd$
- 2) $0 < a^n < b^n$
- 3) $\frac{1}{a} > \frac{1}{b} > 0$
- 4) $\frac{1}{a^n} > \frac{1}{b^n} > 0$

Theorem 4.1.5: For any real numbers a, b, c , and d where $0 < a \leq b$, $0 < c \leq d$, and n is any positive integer:

- 1) $0 < ac \leq bd$
- 2) $0 < a^n \leq b^n$
- 3) $\frac{1}{a} \geq \frac{1}{b} > 0$
- 4) $\frac{1}{a^n} \geq \frac{1}{b^n} > 0$

Theorem 4.1.6: For any real numbers a :

- 1) $a^2 \geq 0$
- 2) $a^2 = 0$ if and only if $a = 0$

Theorem 4.1.7: For any real numbers a and b , if $a < b$ then $a < \frac{a+b}{2} < b$.

Example 4.1.1: Let x and y be any positive real numbers, prove that if $x < y$ then $\frac{x+y}{y+1} < \frac{x+y}{x+1}$.

Example 4.1.2: Let x and y be any positive real numbers where $0 < x < y$, prove that

$$0 < \frac{2x}{y} < 1 + \frac{y^2}{x^2}.$$

Example 4.1.3: Let a and b be any real number where $a + 3 \leq b + 1 < 0$, prove that $2ab \leq ab^2 - a^2b$.

4.2 Solving Inequalities without Absolute Values

Solving inequality is to find the solution of the variable that appears in the inequality. The solution is the real number which satisfies the inequality. The inequality can have more than one solution which we call the set of the solutions as the ***solution set of the inequality***.

4.2.1 Solving the linear inequality

Linear inequality is the inequality in the form $ax + b < cx + d$ (or $ax + b \leq cx + d$) where a , b , c , and d are constants and a and b cannot be zero at the same time. We consider linear inequality in two cases:

- 1) Linear inequality in the form of $ax + b < cx + d$ (or $ax + b \leq cx + d$) where a , b , c , and d are constants and $a \neq c$. We solve the linear inequality by rearrange the inequality to have the constant on one side and the coefficient of the variable is 1.

Example 4.2.1 Find the solution set of the following inequalities.

(a) $3x - 1 \geq x + 5$

(b) $2 - x < 2x - 4$

(c) $\frac{x}{2} + 1 \leq x - 5$

(d) $x + 1 > 5x - 11$

- 2) Linear inequality in the form of $ax + b < ax + c$ (or $ax + b \leq ax + c$) where a , b , and c are constants and $a \neq 0$. We solve this type of linear inequality by considering the value of b and c if whether they satisfy the inequality or not. That is if $b < c$ (or $b \leq c$) then the solution set of the inequality $ax + b < ax + d$ (or $ax + b \leq cx + d$) is the set of real number ($\mathbb{R}$) because the inequality is true for all values of x . If $b \geq c$ (or $b > c$) then the solution set of the inequality $ax + b < ax + d$ (or $ax + b \leq cx + d$) is an empty set ($\emptyset$) because there is no value of x that satisfy the inequality.

Example 4.2.2 Find the solution set of the following inequalities.

(a) $x + 4 \leq x - 3$

(b) $2 - 3x > -2 - 3x$

(c) $4x - 1 < 4x - 1$

(d) $3 + \frac{2x}{3} \geq \frac{2x}{3} - 2$

4.2.2 Solving the polynomial inequalities

Polynomial function is the function of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$ where $a_0, a_1, a_2, \dots, a_n$ are constants and n is a positive integer or zero and n is the degree of the polynomial. The procedures of solving the polynomial inequalities are as following:

- 1) Rearrange the inequality to have zero on one side.
- 2) Write the polynomial in the form of linear factors. If the polynomial cannot be written in the form of linear factors then consider whether the polynomial satisfy the inequality or not. If it does then we conclude that the solution set is a set of real numbers otherwise we conclude that the solution set is an empty set.
- 3) Find the value of the variable which makes the linear factors equal to zero. These values are called the **critical values**.
- 4) Find the value of the variable that satisfies the inequality by considering the real number on the intervals near the critical values.

Note1: The polynomial with degree two, $ax^2 + bx + c$, can be factored when $b^2 - 4ac \geq 0$.

Note2: If $b^2 - 4ac < 0$, $\begin{cases} \text{If } a > 0 \text{ then } ax^2 + bx + c > 0 \\ \text{If } a < 0 \text{ then } ax^2 + bx + c < 0 \end{cases}$

Example 4.2.3 Find the solution set of the inequality $2x^2 - 5x \leq 3$.

Example 4.2.4: Solve the inequality $x^2 + 8 > 6x$.

Example 4.2.5: Find the solution set of the inequality $x^3 + x^2 \geq 2x$

Example 4.2.6: Find the solution set of the inequality $(x^2 + 5x + 4)(2x^2 - x + 3) < 0$.

Example 4.2.7: Find the solution set of the inequality $(-2x^2 - x - 4)(x^2 + 2x + 1) \geq 0$.

4.2.3 Solving the inequalities involving rational functions

Rational function is the function in the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$. The method of solving the inequalities involving rational functions is similar to the method of solving polynomial inequalities.

Example 4.2.3.1: Solve the inequality $\frac{1}{x+4} < \frac{1}{x}$.

Example 4.2.3.2: Find the solution set of the inequality $\frac{x^3 - x^2}{x+1} + 2x > 0$.

Example 4.2.3.3: Solve the inequality $\frac{x^2}{-x^2 + x - 4} \leq -1$.