MA332
Exercise III

1. Compute an LU factorization for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$

Determine a set of basic variables and a set of free variables, and find the general solution to $Ax = 0$. Write it in a form similar to (1). What is the rank of A?

2. Carry out the same steps, with b_1, b_2, b_3, b_4 on the right side, for the transposed matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix}.$$

Write the general solution to

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

as the sum of a particular solution to $Ax = b$ and the general solution to $Ax = 0$,

3. Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution, if

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}?$$

Find two vectors x in the nullspace of A, and the general solution to $Ax = b$.

4. Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank of A?

5. What multiple of row 2 of A will elimination subtract from row 3?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

What will be the pivots? Will a row exchange be required?

6. Which values of a, b, c lead to row exchanges, and which make the matrices singular?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} c & 2 \\ 6 & 4 \end{bmatrix}.$$

7. Suppose A is the 4 by 4 identity matrix except for a vector v in column 2:

$$A = \begin{bmatrix} 1 & v_1 & 0 & 0 \\ 0 & v_2 & 0 & 0 \\ 0 & v_3 & 1 & 0 \\ 0 & v_4 & 0 & 1 \end{bmatrix}.$$

(a) Factor A into LU, assuming $v_2 \neq 0$.
(b) Find A^{-1}, which has the same form as A.

8. Use the Gauss-Jordan method to invert

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

9. By locating the pivots, find a basis for the column space of

$$U = \begin{bmatrix} 0 & 1 & 4 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Express each column that is not in the basis as a combination of the basic columns. Find also a matrix A with this echelon form U, but a different column space.

10. Describe geometrically the subspace of $\mathbf{R}^3$ spanned by
(a) (0, 0, 0), (0, 1, 0), (0, 2, 0);
(b) (0, 0, 1), (0, 1, 1), (0, 2, 1);
(c) all six of these vectors. Which two form a basis?
(d) all vectors with positive components.

11. To decide whether b is in the subspace spanned by $w_1, \ldots, w_l$, let the vectors w be the columns of A and try to solve $Ax = b$. What is the result for
(a) $w_1 = (1, 1, 0)$, $w_2 = (2, 2, 1)$, $w_3 = (0, 0, 2)$, $b = (3, 4, 5)$;
(b) $w_1 = (1, 2, 0)$, $w_2 = (2, 5, 0)$, $w_3 = (0, 0, 2)$, $w_4 = (0, 0, 0)$, and any b?

12. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does
(a) A have a two-sided inverse: $AA^{-1} = A^{-1}A = I$?
(b) $Ax = b$ have *infinitely many* solutions for *every* b?

13. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbf{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

14. (a) Find a basis for the space of all vectors in $\mathbf{R}^6$ with $x_1 + x_2 = x_3 + x_4 = x_5 + x_6$.
(b) Find a matrix with that subspace as its nullspace.
(c) Find a matrix with that subspace as its column space.

15. Find a basis for the following subspaces of $\mathbf{R}^4$:
(a) The vectors for which $x_1 = 2x_4$
(b) The vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$
(c) The subspace spanned by (1, 1, 1, 1), (1, 2, 3, 4), and (2, 3, 4, 5).

21. Do the vectors (1, 1, 3), (2, 3, 6), and (1, 4, 3) form a basis for $\mathbf{R}^3$?