Partial Fractions Decomposition

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are both polynomials. A rational expression is proper if the degree of the numerator is lower than the degree of the denominator; otherwise it is improper. An improper rational expression is the sum of a polynomial and a proper rational expression. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

For example

$$\frac{x^3 + x}{x + 2} = x^2 - 2x + 5 - \frac{10}{x + 2}$$

where $x^2 - 2x + 5$ is the quotient and -10 is the remainder upon division of the denominator into the numerator.

Partial fraction decomposition is the process of rewriting a rational expression as the sum of a quotient polynomial plus partial fractions. If the rational expression is proper, the quotient will be zero.

Example 1: Find the partial fraction decomposition of $\frac{x^3 + 2x^2 - 1}{x - 2}$.

After finding the quotient, the next step in decomposing a rational expression is factoring the denominator. It can be shown that any polynomial Q (the denominator) can be factored as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form ax^2+bx+c, where $b^2-4ac<0$). For instance, if $Q(x)=x^4-16$, we could factor it as

$$Q(x)=x^4-16=\left(x^2+4\right)\left(x^2-4\right)=\left(x^2+4\right)(x+2)(x-2)$$

The third step is to express the proper rational function $R(x)/Q(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{\left(ax^2+bx+c\right)^j}$$

<u>CASE I</u>: The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, ..., A_k$ such that

$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k} \qquad (1)$$

Example 2: Find the partial fraction decomposition of

(a) $\dfrac{x-9}{x^2+3x-10}$

(b) $\dfrac{3x^2-7x-2}{x^3-x}$

<u>CASE II</u>: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ in Equation (1), we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r} \qquad (2)$$

By way of illustration, we could write

$$\frac{x^3+4}{x^2(x+1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}$$

Example 3: Find the partial fraction decomposition of

(a) $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

(b) $\dfrac{x-2}{x^2(x-1)^2}$

<u>CASE III</u>: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then, in addition to the partial fractions in Equation (1) and (2), the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c} \tag{3}$$

where A and B are constants to be determined. For instance, the function given by $x/\left[(x-3)(x^2+1)(x^2+2)\right]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-3)(x^2+1)(x^2+2)}=\frac{A}{x-3}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+2}$$

Example 4: Find the partial fraction decomposition of

(a) $\dfrac{3x^2-4x+3}{x^3+x}$

(b) $\dfrac{6x^2-3x+1}{(4x+1)(x^2+1)}$

<u>CASE IV</u>: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction (3), the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r} \qquad (4)$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the term in (4) can be integrated by first completing the square.

Example 5: Find the partial fraction decomposition of $\frac{6x^2-15x+22}{(x+3)(x^2+2)^2}$.