Optimization without Constraints: One Independent Variable II

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CW Ch.9

Outline

- Maclaurin and Taylor Series
- Nth-Derivative Test for Relative Extremum of a Function of One Variable

Maclaurin Series Expansion

- To expand a function $y = f(x)$ around a point x_0 means to transform that function into a polynomial form, in which the coefficients of the various terms are expressed in terms of the derivative values evaluated at the point x_0.
- Consider a polynomial function of nth degree,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

- We can get the derivatives as follows:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$$

$$f''(x) = 2a_2 + 3(2)a_3 x + 4(3)a_4 x^2 + \cdots + n(n-1)a_n x^{n-2}$$

$$f'''(x) = 3(2)a_3 + 4(3)(2)a_4 x + \cdots + n(n-1)(n-2)a_n x^{n-3}$$

$$\vdots$$

$$f^{(n)}(x) = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)a_n$$

Maclaurin Series Expansion

- We evaluate these derivatives at $x = 0$,

$$f'(0) = a_1$$

$$f''(0) = 2a_2$$

$$f'''(0) = 3(2)a_3$$

$$\vdots$$

$$f^{(n)}(0) = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)a_n$$

- Let $n! \equiv n(n-1)(n-2)\cdots(3)(2)(1)$, where n is a positive integer and $0! = 1$.
- We can write

$$a_1 = \frac{f'(0)}{1!}, a_2 = \frac{f''(0)}{2!}, a_3 = \frac{f'''(0)}{3!}, \ldots, a_n = \frac{f^{(n)}(0)}{n!}$$

Maclaurin Series Expansion

- We can not express the given function $f(x)$ as:

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

- This new polynomial, called the Maclaurin series of the polynomial function $f(x)$, represents the expansion of the function $f(x)$ around zero.
- Example: $f(x) = 2 + 4x + 3x^2$

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$$

$$f(x) =$$

Taylor Series Expansion

- More generally, the polynomial function $f(x)$ can be expanded around any point x_0, not necessarily zero.
- In general, ***Taylor formula*** can be written as:

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

- Observe that if we set $x_0 = 0$, Taylor series expansion will become exactly the same as Maclaurin series expansion.
- Example: $f(x) = 2 + 4x + 3x^2$

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$f(x) =$$

Expansion of an Arbitrary Function

- According to a mathematical proposition known as Taylor's Theorem, given an arbitrary function $\phi(x)$, this function can be expanded around the point $x = x_0$ as follows:

$$\phi(x) = \left[\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n\right] + R_n$$

- The appearance of R_n is due to the fact that we are dealing with an arbitrary function ϕ which cannot always be transformed exactly into the polynomial form.
- The first part can be thought of an a polynomial approximation of the function, and R_n can be thought of as the error term.
- We can choose n to be any value. For example, if we choose $n = 1$, it will constitutes a linear approximation to $\phi(x)$. If we choose $n = 2$, it will constitutes a quadratic approximation to $\phi(x)$.
- $R_n = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}$, where p is close to x_0.

Example

- Let $\phi(x) = 5 + 2x + x^2$. We will expand the function around $x_0 = 1$, with $n = 1$.

$$\phi(x_0) = \phi(1) = 5 + 2 + 1 = 8$$
$$\phi'(x_0) = \phi'(1) = 2 + 2(1) = 4.$$

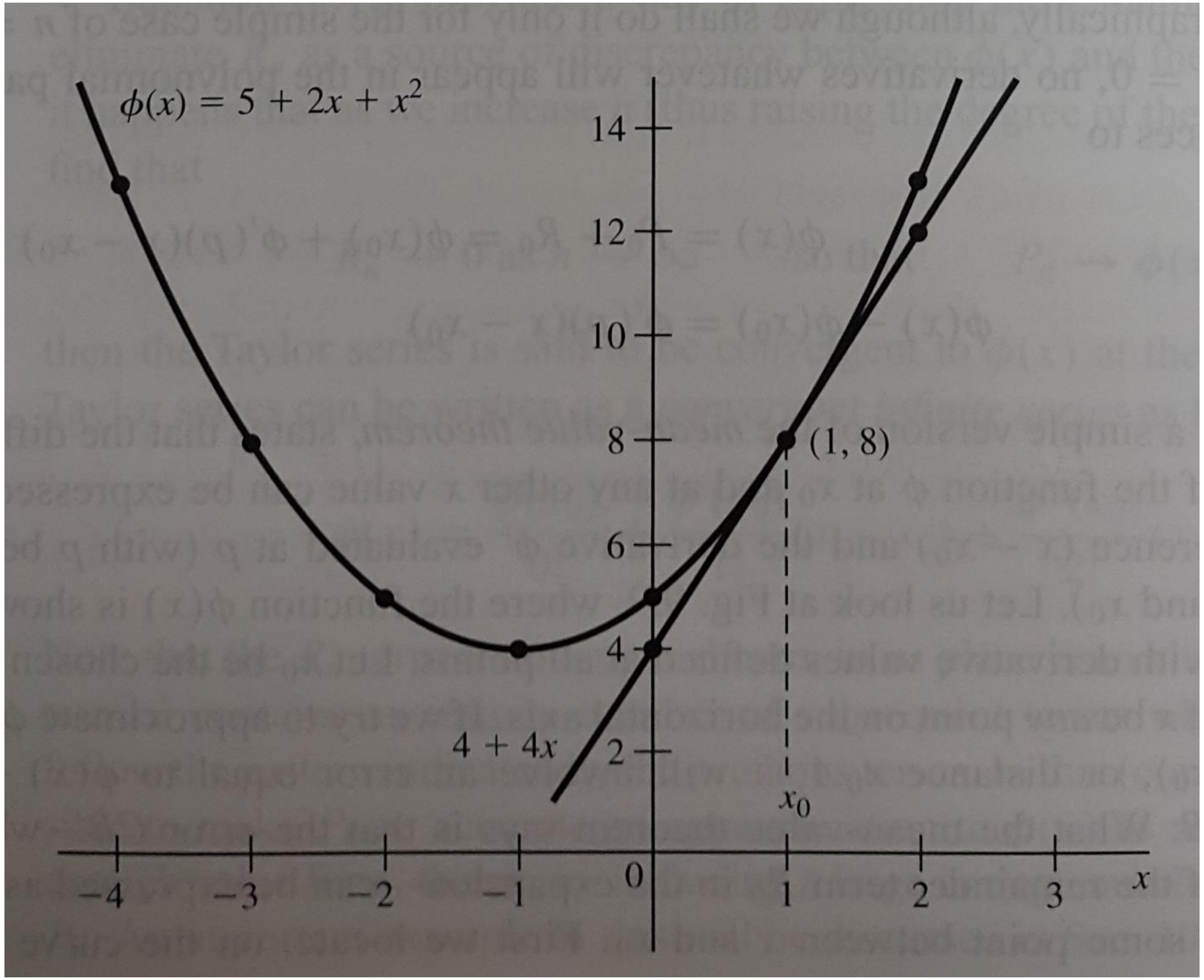
From Taylor's formula with remainder gives us

$$\begin{aligned} \phi(x) &= \phi(x_0) + \phi'(x_0)(x - x_0) + R_1 \\ &= 8 + 4(x - 1) + R_1 \\ &= 4 + 4x + R_1, \end{aligned}$$

where $4 + 4x$ is a linear approximation and the R_1 term represents the error of the (linear) approximation.

Example (cont.)

- Let $\phi^{(A)}(x) = 4 + 4x$. We can see that
  - $\phi(0)$ =5, $\phi^{(A)}(0) = 4$
  - $\phi(1)$ =8, $\phi^{(A)}(1) = 8$
  - $\phi(2)$ =13, $\phi^{(A)}(2) = 12$.
- We approximate a quadratic function with a linear function.
- This works well just in the close neighborhood of the point x_0.



*N*th-Derivative Test for Relative Extremum of a Function of One Variable

- What can we do if the second derivative of a function is zero?
- An interesting observation is that:

  "A function $f(x)$ attains a relative maximum (minimum) value at x_0 if $f(x) - f(x_0)$ is negative (positive) for values of x in the immediate neighborhood of x_0, both to its left and to its right."

- From Taylor's formula, we can write:

$$f(x) - f(x_0) = +\frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n.$$

  And from

$$R_n = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1},$$

  where p is close to x_0.

*N*th-Derivative Test for Relative Extremum of a Function of One Variable

- That is we can write:

$$f(x) - f(x_0) = +\frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}.$$

- **Case 1**: $f'(x_0) \neq 0$ and $n = 0$.

$$f(x) - f(x_0) = \frac{f'(p)}{(1)!}(x - x_0).$$

Suppose $f'(x_0) < 0$, $(x - x_0) < 0$ implies $f(x) - f(x_0) > 0$, and $(x - x_0) > 0$ implies $f(x) - f(x_0) < 0$. $f(x_0)$ is not at a relative extremum.

- **Case 2**: $f'(x_0) = 0$; $f''(x_0) \neq 0$ and $n = 1$.

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(p)}{(2)!}(x - x_0)^2 = \frac{f''(p)}{2}(x - x_0)^2.$$

Suppose $f''(x_0) < 0$, $f(x) - f(x_0) < 0$ for any value of $x \neq x_0$. This means $f(x_0)$ is a relative maximum.

*N*th-Derivative Test for Relative Extremum of a Function of One Variable

- That is we can write:

$$f(x) - f(x_0) = +\frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}.$$

- **Case 3**: $f'(x_0) = f''(x_0) = 0$; but $f'''(x_0) \neq 0$ and $n = 2$.

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(x_0)}{(2)!}(x - x_0)^2 + \frac{f'''(p)}{(3)!}(x - x_0)^3 = \frac{f'''(p)}{6}(x - x_0)^3.$$

  Suppose $f'''(x_0) < 0$, $(x - x_0) < 0$ implies $f(x) - f(x_0) > 0$, and $(x - x_0) > 0$ implies $f(x) - f(x_0) < 0$. $f(x_0)$ is not at a relative extremum. (But it should be an inflection point.)

- **Case 4**: $f'(x_0) = f''(x_0) = f'''(x_0) = 0$; but $f^{(4)}(x_0) \neq 0$ and $n = 3$.

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(x_0)}{(2)!}(x - x_0)^2 + \frac{f'''(x_0)}{(3)!}(x - x_0)^3 + \frac{f^{(4)}(p)}{(4)!}(x - x_0)^4 = \frac{f^{(4)}(p)}{(4)!}(x - x_0)^4.$$

  Suppose $f^{(4)}(x_0) < 0$, $f(x) - f(x_0) < 0$ for any value of $x \neq x_0$. This means $f(x_0)$ is a relative maximum.

*N*th-Derivative Test for Relative Extremum of a Function of One Variable

- In conclusion, if the first derivative of a function $f(x)$ at x_0 is $f(x_0) = 0$ and if the first *nonzero* derivative value at x_0 encountered in successive derivation is that of the Nth derivative, $f^{(N)}(x_0) \neq 0$, then the value $f(x_0)$ will be:
  a) A relative ***maximum*** if N is an even number and $f^{(N)}(x_0) < 0$
  b) A relative ***minimum*** if N is an even number but $f^{(N)}(x_0) > 0$
  c) An ***inflection point*** if N is an odd number.

Example

- Given the function $y = f(x) = (7 - x)^4$.

FOC $\quad f'(x) = -4(7 - x)^3 = 0$

$$x^* = 7$$

Higher-order derivative test:

$$f''(x) = 12(7 - x)^2, \qquad \text{so that } f''(7) = 0$$

$$f'''(x) = -24(7 - x), \qquad \text{so that } f'''(7) = 0$$

$$f^{(4)}(x) = 24, \qquad \text{so that } f^{(4)}(7) = 24.$$

Since $N = 4$ is an even number, and $f^{(4)}(7) > 0$, we conclude that $f(7)$ is a relative *minimum*.

Applications

- **Monopolist's Profit Maximization**

  Let the average-revenue (or demand) function be

  $$AR(Q) = 1000 - 20Q - Q^2,$$

  and the cost function be

  $$C(Q) = Q^3 - 2Q^2 + 40Q + 20$$

  For a monopolist,

  $$TR(Q) = AR(Q) \cdot Q = 1{,}000Q - 20Q^2 - Q^3.$$

  Then, the profit function is

  $$\pi(Q) = TR(Q) - C(Q) = -2Q^3 - 18Q^2 + 960Q - 20$$

Applications

- **Monopolist's Profit Maximization**

FOC $\pi'(Q) = -6Q^2 - 36Q + 960 = 0$

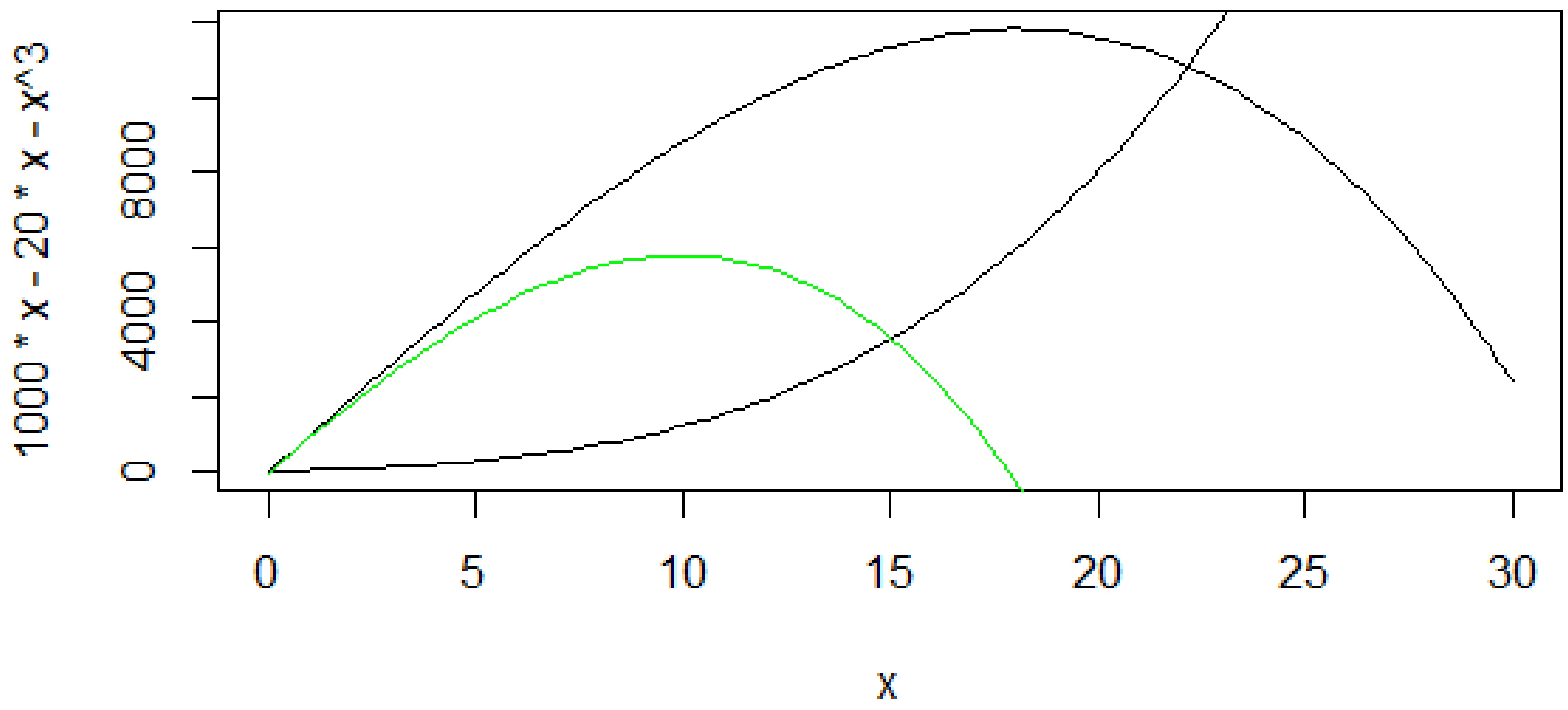
$$Q^* = -16,10$$

SOC $\pi''(Q) = -12Q - 36.$

At $Q^* = 10, \quad \pi''(10) = -156 < 0.$

We have the profit-maximizing output level $Q^* = 10$ and the maximum profit at $\pi(10) = -2{,}000 - 1{,}800 + 9600 - 20 = 5{,}780$ baht.

Note: We can solve this through the profit-maximizing condition MR=MC as well.


1000 * x - 20 * x - x^3
0
4000
8000
0
5
10
15
20
25
30
x

Applications

- **Coefficients of a Cubic Total-Cost Function**

  Total cost function must be strictly increasing. As we produce more, the cost of production must be higher.

  However, some cubic functions may have downward-sloping segment. We are now trying to restrict the coefficient values of the cubic functions to make them strictly increasing.

  Given a general cubic cost function,

  $$C(Q) = aQ^3 + bQ^2 + cQ + d.$$

  We have

  $$MC(Q) = C'(Q) = 3aQ^2 + 2bQ + c.$$

Applications

- **Coefficients of a Cubic Total-Cost Function**

  To make the cost function strictly increasing, MC must be strictly positive. We can make sure of this by setting $a > 0$.

  Then, the minimum point of MC must also be higher than 0.

  To figure out the minimum point,

  FOC $\quad MC'(Q) = 6aQ + 2b = 0$

  $$Q^* = -\frac{b}{3a}.$$

  SOC $\quad MC''(Q) = 6a > 0$ ➔ Relative Minimum.

  Then, $MC_{min} = MC\left(-\frac{b}{3a}\right) = 3\text{a}\left(-\frac{b}{3a}\right)^2 + 2b\left(-\frac{b}{3a}\right) + c = \frac{3ac-b^2}{3a}.$

  We must have $MC_{min} > 0$, or $b^2 < 3ac$.

Assignments

- Exercise 9.4 No. 2, 3, 5.