Solution: Exercise 4

Application of the Derivative: Related Rates, Linearization & Differentials

1 Linearization & Differentials

1. Determine the linear approximation for $\sin(\theta)$ at $\theta = 0$.

   Solution: From $f(\theta) = \sin(\theta)$, $f'(\theta) = \cos(theta)$ and $f(0) = \sin(0) = 0$, $f'(0) = \cos(0) = 1$. The linearization of $f(x)$ at $x = 0$ is given by

$$L(\theta) = f(0) + f'(0)(\theta - 0) = 0 + 1(\theta - 0) \quad \Longrightarrow \quad L(\theta) = \theta.$$

2. Find a linearization of the given function at the indicated number.

   (a) $f(x) = \ln(x)$ at $x = 1$.
   Solution: From $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$ and $f(1) = \ln(1) = 0$, $f'(1) = 1$. The linearization of $f(x)$ at $x = 1$ is given by

$$L(x) = f(1) + f'(1)(x - 1) = 0 + 1(x - 1) \quad \Longrightarrow \quad L(x) = x - 1.$$

   (b) $f(x) = x + e^{x-5}$ at $x = 5$
   Solution: From $f(x) = x + e^{x-5}$, $f'(x) = 1 + e^{x-5}$ and $f(5) = 5 + e^0 = 6$, $f'(1) = 1 + e^0 = 2$. The linearization of $f(x)$ at $x = 5$ is given by

$$L(x) = f(1) + f'(1)(x - 1) = 6 + 2(x - 1) \quad \Longrightarrow \quad L(x) = 2x - 4.$$

3. Find an approximation of each given quantity by using a local linear approximation.

   (a) $(8.1)^{2/3}$

   Solution:
   Let $f(x) = x^{2/3}$. Then $(8.1)^{2/3} = f(8.1)$. We want use the linearization at a point that is easy to evaluate f and close to -0.9. Here we will use $x = 8$. The linearization of $f(x)$ at $x = 8$ is given by

$$L(x) = f(8) + f'(8)(x - 8).$$

   Since $f'(x) = \frac{2}{3}x^{-1/3}$, then $f(8) = x^{2/3} = 4$ and $f'(x) = \frac{2}{3}8^{-1/3} = \frac{1}{3}$ which gives

$$L(x) = 4 + \frac{1}{3}(x - 8).$$

   Therefore, a local linear approximation of $(8.1)^{2/3} = f(8.1)$ is
   $L(8.1) = 4 + \frac{1}{3}(8.1 - 8) = 4 + 1/30 = \frac{121}{30}$.

   (b) $\frac{(-0.9)^3}{1+(-0.9)^2}$

   Solution:

Let $f(x) = \frac{x^3}{1+x^2}$. Then $\frac{(-0.9)^3}{1+(-0.9)^2} = f(-0.9)$. We want use the linearization at a point that is easy to evaluate f and close to -0.9 Here we will use $x = -1$. The linearization of $f(x)$ at $x = -1$ is given by

$$L(x) = f(-1) + f'(-1)(x - (-1)).$$

Since $f'(x) = \frac{3x^2(1+x^2)-x^3(2x)}{(1+x^2)^2} = \frac{2x^4+3x^2}{(1+x^2)^2}$, then $f(-1) = \frac{(-1)^3}{1+(-1)^2} = -\frac{1}{2}$ and $f'(-1) = \frac{2(-1)^4+3(-1)^2}{(1+(-1)^2)^2} = \frac{5}{4}$ which gives

$$L(x) = -\frac{1}{2} + \frac{5}{4}(x+1).$$

Therefore, a local linear approximation of $\frac{(-0.9)^3}{1+(-0.9)^2} = f(-0.9)$ is
$L(-0.9) = -\frac{1}{2} + \frac{5}{4}(-0.9+1) = -\frac{1}{2} + \frac{5}{40} = \frac{15}{40}$.

4. Compute the differential dy for $y = x^2\sin(2x)$.

   Solution: By definition, the differential is $dy = \frac{dy}{dx}dx$.
   Since $\frac{dy}{dx} = x^2\frac{d}{dx}\sin(2x) + \sin(2x)\frac{d}{dx}x^2 = 2x^2\cos(2x) + 2x\sin(2x)$,

$$dy = [2x^2\cos(2x) + 2x\sin(2x)]dx.$$

5. Compute dy and $\triangle y$ for $y = \frac{1}{\sqrt{x}}$ as x changes 4 from to 4.02 .

   Solution:

   Let $f(x) = \frac{1}{\sqrt{x}}$. We are given that $dx = \triangle x = 4.02 - 4 = 0.02$. Note: $f'(x) = \frac{dy}{dx} = -\frac{1}{2}x^{-3/2}$, $f'(4) = -\frac{1}{16}$

$$dy = f'(4)dx = -\frac{1}{16}\cdot 0.02 = -\frac{1}{800}$$

   and

$$\triangle y = f(4.02) - f(4) = \frac{1}{\sqrt{4.02}} - \frac{1}{2}.$$

6. Consider the function

$$f(x) = \frac{1}{(1+2x)^3}.$$

   (i) Find a local linear approximation of f at $x = 0$.
   Solution: $f'(x) = \frac{1}{(1+2x)^3} = -6(1+2x)^{-4}$. Since $f(0) = 1$ and $f'(0) = -6$, A local linear approximation of $f(x)$ at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x-0) \quad \Longrightarrow \quad L(x) = 1 - 6x.$$

   (ii) Use (i) to approximate $(1.1)^{-3}$.
   Solution: We want $(1.1)^{-3} = \frac{1}{(1+2x)^3} \quad \Rightarrow \quad 1.1 = 1 + 2x \quad \Rightarrow \quad x = (1.1-1)/2 = 0.05$. Therefore,

$$(1.1)^{-3} = f(0.05) \approx L(0.05) = 1 - 6(0.05) = 1 - 0.3 = 0.7.$$

   That is, an approximation of $(1.1)^{-3}$ by a local linear approximation in (i) is 0.7.

7. The area of a circle with radius r is $A = \pi r^2$.

   (i) If the radius of a circle changes from 4 cm to 5 cm, find the exact change in the area.
   Solution: The exact change is $|\pi 5^2 - \pi 4^2| = 9\pi$.

   (ii) By using the *approximation from differentials*, what is the approximate change in the area?
   Solution: Let $A(r) = \pi r^2$ be a function of the area in term of radius r.

$$dA = A'(r)dr = 2\pi r dr = 2\pi(4)(5-4) = 8\pi,$$

   where we have used $dr = 5 - 4 = 1$.

2 Related Rates

1. Let $s(t) = 2\sqrt{t} + \cos(\pi t)$ be the position function of an object that moves on a horizontal line. Find the velocity and acceleration functions.

   Solution:

   - The velocity function is $v(t) = \frac{d}{dt}s(t) = t^{-1/2} - \pi\sin(\pi t)$.
   - The acceleration function is $a(t) = \frac{d}{dt}v(t) = -\frac{1}{2}t^{-3/2} - \pi^2\cos(\pi t)$.

2. Let $s(t) = 2t^3 - 12t^2 + 48$ be the position function of an object that moves on a horizontal line.

   (a) Find the velocity function $v(t)$ and the acceleration function $a(t)$.
   (b) Find the position of the object when $a(t) = 0$.

   Solution:

   (a) – The velocity function is $v(t) = 6t^2 - 24t$.
       – The acceleration function is $a(t) = \frac{d}{dt}v(t) = 12t - 24$.

   (b) When $a(t) = 0$,

$$a(t) = 12t - 24 = 0 \quad \Longrightarrow \quad t = \frac{24}{12} = 2.$$

   Therefore, the position when $a(t) = 0$ is $s(2) = 2(2^3) - 12(2^2) + 48 = 16$.

3. A tank in the shape of a right circular cylinder of radius 8 m is being filled with water at a constant rate of 10 m^3/min. How fast is the level of the water rising? (Or at what rate is the height of the water increasing?)

   Solution:

   Let V and h be the volume and the height, respectively, of the water being filled in the right circular cylinder of radius r. The volume V is given in terms of r and h by

$$V = \pi r^2 h.$$

   We are given the rate of water being filled:

$$\frac{dV}{dt} = 10\ m^3/\text{min}$$

and we want to find the rate at which the height is increasing: $\frac{dh}{dt}$. Form above,

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt} = \frac{1}{64\pi}10 = \frac{5}{32\pi} \text{ m/min} .$$

4. A plate in the shape of an equilateral triangle expands with time. The length of a side increases at a constant rate of 2 cm/hr. At what rate is the area increasing when the side is 8 cm?

   Solution: Let x be the length of a side of the equilateral triangle . The area A of the equilateral triangle is given in terms of x by

$$A = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} x\sqrt{x^2 - (x/2)^2} \quad \Longrightarrow \quad A = \frac{\sqrt{3}}{4}x^2.$$

   We are given that the length of a side increases at a constant rate

$$\frac{dx}{dt} = 2 \text{ cm/hr}.$$

   and we want to find the rate at which the area is increasing: $\frac{dA}{dt}$ when $x = 8$. Form above,

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}\left(\frac{d}{dt}x^2\right) \quad \Longrightarrow \quad \frac{dA}{dt} = \frac{\sqrt{3}\ x}{2}\frac{dx}{dt} = \frac{\sqrt{3}\ \ 8}{2}\ 2 = 8\sqrt{3} \text{ cm}^2\text{/hr}.$$

5. Ship S_1 sails north and passes a dock with a constant rate of 10 miles per hour at noon. Ship S_2 sails west and passes the same dock at a constant rate of 15 miles per hour at 1:00pm. At what rate is the distance between the two ships changing at 2:00pm?

Solution:
Let y be the distance of Ship S_1 from the dock, x be the distance of Ship S_2 from the dock, and z be the distance between S_1 and S_2. Since Ship S_1 sails north from the dock and Ship S_2 sails west from the dock, we can use the Pythagorean theorem to obtain

$$z^2 = x^2 + y^2.$$

We are given that Ship S_1 sails north with a constant rate of 10 miles per hour (mi/hr)and Ship S_2 sails west with a constant rate of 15 miles, i.e.

$$\frac{dy}{dt} = 10 \text{ mi/hr} \qquad \frac{dx}{dt} = 15 \text{ mi/hr} .$$

We want to find the rate at which the distance between the two ships is changing at 2:00pm, i.e.

$$\text{At 2pm, find } \frac{dz}{dt}.$$

At 2pm, we have that

$$y = 10 \times (2 - 0) = 20 \text{ miles} \qquad x = 15 \times (2 - 1) = 15 \text{ miles}, \qquad z = \sqrt{20^2 + 15^2} = 25 \text{ miles}.$$

From above, differentiating both sides with respect to t (using Implicit Differentiation) gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Longrightarrow \quad z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Therefore, at 2pm, by substituting $x = 15$, $y = 20$, $\frac{dx}{dt} = 15$, $\frac{dy}{dt} = 10$, we can solve for $\frac{dz}{dt}$:

$$25\frac{dz}{dt} = (15)(15) + (20)(10) \qquad \Longrightarrow \qquad \frac{dz}{dt} = \frac{225 + 200}{25} = 17 \text{ mi/hr} .$$