

Relations & Functions II

TU152: Fundamental Mathematics

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Definition (Function)

A function f from a set X to a set Y , denoted $f : X \rightarrow Y$, is a relation from X to Y that satisfies the following two properties.

- (1) Every element in X is related to some element in Y . I.e.

$$\forall x \in X, \exists y \in Y, \text{ such that } (x, y) \in f.$$

- (2) No element in X is related to more than one element in Y . I.e.

$$\forall x \in X, \text{ if } (x, y_1), (x, y_2) \in f \text{ where } y_1, y_2 \in Y, \text{ then } y_1 = y_2$$

The set X is called the **domain** of f and Y is called the **co-domain** of f .

- The set of all values of f taken together is called the **range** of f or the **image** of X under f . I.e.,

$$\text{range of } f = \text{image of } X \text{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \in X\}.$$

- Given an element y in Y , there may exist elements in X with y as their image. If $f(x) = y$, then x is called a **preimage** of y or an **inverse image** of y . The set of all inverse images of y is called the **inverse image** of y . I.e.,

$$\text{the inverse image of } y = \{x \in X \mid f(x) = y\}$$

A “function” is not **well defined** if it fails to satisfy at least one of the requirements for being a function.

Function

Let f be a function from a set X to a set Y , denoted $f : X \rightarrow Y$. Let $(x, y) \in f$,

- We say that “ f sends x to y ” or “ f maps x to y ” and
- write $f : x \rightarrow y$ or $f(x) = y$.
- The unique element to which f sends x is denoted $f(x)$ and is called
 - f of x , or
 - the output of f for the input x , or
 - the value of f at x , or
 - the image of x under f .

Arrow Diagrams of Functions

Recall that if X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram. You make a list of elements in X and a list of elements in Y , and draw an arrow from each element in X to the corresponding element in Y . The arrow diagram that defines a function must have the following two properties.

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y .

Function

Equality of Functions

If $F : X \rightarrow Y$ and $G : X \rightarrow Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Definition (Graph)

Let A and B be subsets of $\mathbb{R}$. A function $f : A \rightarrow B$ is called a real-valued function of a real variable. In this case, each ordered pair $(x, f(x))$ can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph of f** .

Definition (Increasing/Decreasing Function)

Let X be the domain of a function f and $S \subset X$.

- f is increasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is decreasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Example:

- (1) Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x - 3$ is increasing on $\mathbb{R}$.
- (2) Show that the function $f : \mathbb{R} - \{-1\} \rightarrow \mathbb{R}$, $f(x) = \frac{x+2}{x+1}$ is decreasing on $(-\infty, -1) \cup (-1, \infty)$.

Examples of Functions

- **The Identity Function on a Set:** Given a set X , define a function I_X from X to X by

$$I_X(x) = x$$

for all x in X .

- **Sequences:** The formal definition of sequences specifies that an infinite sequence is a function defined on the *set of integers that are greater than or equal to a particular integer*. E.g. the sequence denoted

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots, \frac{(-1)^n}{n+1}, \dots$$

can be thought of as the function f from the nonnegative integers to the real numbers that associates

$$0 \rightarrow 1, 1 \rightarrow -\frac{1}{2}, 2 \rightarrow \frac{1}{3}, 3 \rightarrow -\frac{1}{4}, 4 \rightarrow \frac{1}{5}, \dots, n \rightarrow \frac{(-1)^n}{n+1}, \dots$$

That is, $f : \mathbb{Z}^+ \cup \{0\} \rightarrow \mathbb{R}$ is the function defined as follows:

$$f(n) = \frac{(-1)^n}{n+1}.$$

Define a function g from the positive integers to the real numbers for the above sequence, $g : \mathbb{Z}^+ \rightarrow \mathbb{R}$,

$$g(x) =$$

Examples of Functions

- **A Function Defined on a Power Set:** $\mathcal{P}(A)$ denotes the set of all subsets of the set A . Define a function $F : \mathcal{P}(\{a, b, c\}) \rightarrow \mathbb{Z}^+ \cup \{0\}$ as follows. For each $X \in \mathcal{P}(\{a, b, c\})$,

$$F(X) = \text{the number of elements in } X.$$

Draw an arrow diagram for F .

- **Functions Defined on a Cartesian Product:** E.g., define functions

$M : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $R : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ as:

for all ordered pairs (a, b) of real numbers, $M(a, b) = ab$ and $R(a, b) = (-a, b)$.

Find the following:

$$M(-1, -1) =$$

$$R(-1, -1) =$$

$$M(-\sqrt{2}, \sqrt{2}) =$$

Exercise: Explain why each of M and R is indeed a function.

Injective (one-to-one) functions

Definition (Injective or one-to-one functions)

Let F be a function from a set X to a set Y . F is **one-to-one** or **injective** if, and only if, for all elements x_1 and x_2 in X ,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently (by contrapositive),

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

A function $F : X \rightarrow Y$ is **not** one-to-one if, and only if,

$$\exists x_1, x_2 \in X \text{ with } F(x_1) = F(x_2) \text{ and } x_1 \neq x_2.$$

In terms of arrow diagrams,

- A one-to-one function takes **distinct** points of the domain to **distinct** points of the co-domain.
- A function is not one-to-one will have at least two points of the domain taken to the same point of the co-domain.

Example: Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H : X \rightarrow Y$ as follows:

$H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K : X \rightarrow Y$ as follows:

$K(1) = d$, $K(2) = b$, and $K(3) = d$. Determine if each of the functions H and K is one-to-one.

One-to-One Functions on Infinite Sets

Let f be a function defined on an infinite set X . By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

- To prove f is one-to-one, we will generally use the method of *direct proof*: suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$ and show that $x_1 = x_2$.
- To show that f is **not** one-to-one, we will generally find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Example: Show that the identity function I_X on a set X is injective.

Example: Define $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$ by the rules

$f(x) = 4x - 1$ for all $x \in \mathbb{R}$, and $g(n) = n^2$ for all $n \in \mathbb{Z}$.

- 1 Is f one-to-one? Prove or give a counterexample.
- 2 Is g one-to-one? Prove or give a counterexample.

Example: Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing then f is one-to-one.

Function

Definition (Onto or Surjective Functions)

Let F be a function from a set X to a set Y . F is onto (or surjective) if, and only if, given any element y in Y , it is possible to find an element x in X with the property that $y = F(x)$. $F : X \rightarrow Y$ is onto if, and only if

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

A function is not onto if, and only if,

$$\exists y \in Y \text{ such that } \forall x \in X, F(x) \neq y,$$

i.e., there is some element in Y that is not the image of any element in X .

Example:

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$.

Define $H : X \rightarrow Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$.

Define $K : X \rightarrow Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$.

Determine whether each of the functions H and K are surjective (onto) or not.

Function

Example

- 1 Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x - 5$ is surjective.
- 2 Show that the function $g : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $g(n) = 3n - 5$ is not surjective.

Example Let A and B be two nonempty sets.

The functions $P_A : A \times B \rightarrow A$ defined by $P_A(a, b) = a$ and $P_B : A \times B \rightarrow B$ defined by $P_B(a, b) = b$ are called projection functions. Show that P_A and P_B are surjective functions.

Example(Continued)

Function

Definition (one-to-one correspondence or bijection)

A **one-to-one correspondence** or **bijection** from a set X to a set Y is a function $F : X \rightarrow Y$ that is both *one-to-one* and *onto*.

Example

- 1 Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x - 5$ is a bijective function.
- 2 Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is not bijective.

Example (A Function of Two Variables): Define a function $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ as :

$$\text{For all } (x, y) \in \mathbb{R} \times \mathbb{R}, F(x, y) = (x + y, x - y).$$

Is F a one-to-one correspondence from $\mathbb{R} \times \mathbb{R}$ to itself?

Example (Continued)

Inverse of A Function Vs. Inverse Function

In general, for any function f , it is possible to define an inverse map as follows.

Inverse of A Function

Suppose $f : X \rightarrow Y$ is a function. Then we can write

$$y = f(x), x \in X \quad \text{or} \quad f = \{(x, y) | x \in X, y = Y\}.$$

The **inverse of function** f , denoted f^{-1} , is defined as

$$\boxed{f^{-1}(y) = x \Leftrightarrow y = f(x)} \quad \text{or} \quad \boxed{f^{-1} = \{(y, x) | (x, y) \in f\}}.$$

Note that the inverse of function f may or may not be a function. If f^{-1} is a function f^{-1} is called the **inverse function** of f .

Note that, the inverse of a function may or may not be a function.

Inverse Function

Suppose $f : X \rightarrow Y$ is a one-to-one correspondence or bijective (one-to-one and onto).

Then the **inverse of function** f , f^{-1} , is a function and it is called the **inverse function of f** . As before, $f^{-1} : Y \rightarrow X$ is defined as:

$$\boxed{f^{-1}(y) = x \Leftrightarrow y = f(x)} \quad \text{or} \quad \boxed{f^{-1} = \{(y, x) | (x, y) \in f\}}.$$

Example: Let $X = \{1, 2, 3\}$ and $Y = \{-1, -2, -3\}$.

Let $f = \{(1, -3), (2, -1), (3, -3)\}$ be a function from X to Y .

Find the inverse of f , f^{-1} and determine whether it is the inverse function or not.

Example: Let $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 2x - 3\}$ be a function. Find the inverse of the function f and determine whether it is the inverse function of f or not.

Inverse Functions

If F is a one-to-one correspondence from a set X to a set Y , then there is a function from Y to X that “undoes” the action of F . That is, it sends each element of Y back to the element of X that it came from. This function is called the inverse function for F .

Inverse Functions

Suppose $F : X \rightarrow Y$ is a one-to-one correspondence. I.e., suppose F is one-to-one and onto. Then there is a function $F^{-1} : Y \rightarrow X$ that is defined as follows:

Given any element $y \in Y$,

$F^{-1}(y)$ = the unique element $x \in X$ such that $F(x)$ equals y . I.e.,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

The function F^{-1} is called the **inverse function** for F .

Theorem

If X and Y are sets and $F : X \rightarrow Y$ is one-to-one and onto, then $F^{-1} : Y \rightarrow X$ is also one-to-one and onto.

Inverse Functions

Example The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula $f(x) = 4x - 1$ for all real numbers x was shown to be one-to-one in the previous examples. Find its inverse function.

Inverse Functions

Example A function $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ defined as, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$F(x, y) = (x + y, x - y)$$

is shown to be a bijective function in the previous example. Determine its inverse function $F^{-1} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$.