Solution: Assignment 1

1. Determine whether the statement forms are logically equivalent. In each case, construct a truth table to justify your answer.

   (a) $(p \to q) \to (q \to p)$ and $(p \vee q) \to (p \wedge q)$

   (b) $\sim p \to (q \to r)$ and $q \to (p \vee r)$

   (c) $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$

   (d) $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$

Answer:

(a) $(p \to q) \to (q \to p)$ and $(p \vee q) \to (p \wedge q)$

Truth table:

p	q	$p \to q$	$q \to p$	$p \vee q$	$p \wedge q$	$(p \to q) \to (q \to p)$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	F	T
F	F	T	T	F	F	T	F

Since rows 3 and 4 of the truth table for $(p \to q) \to (q \to p)$ and $(p \vee q) \to (p \wedge q)$ have different truth values, then these statement forms are not logically equivalent. ■

(b) $\sim p \to (q \to r)$ and $q \to (p \vee r)$

p	q	r	$\sim p$	$q \to r$	$p \vee r$	$\sim p \to (q \to r)$	$q \to (p \vee r)$
T	T	T	F	T	T	T	T
T	T	F	F	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	T

Since all rows 8 (all possible cases for truth values of p, q, r) in the truth table for $\sim p \to (q \to r)$ and $q \to (p \vee r)$ have the same truth values, then these statement forms are logically equivalent. ■

(c) $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$

Truth table:

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$	$p \leftrightarrow q$
T	T	T	T	F	F	T
T	F	T	F	T	T	F
F	T	T	F	T	T	F
F	F	F	F	T	F	T

Since all 4 rows of the truth table for $(p \vee q) \wedge \sim (p \wedge q)$ and $p \leftrightarrow q$ have different truth values, then these statement forms are not logically equivalent. ■

(d) Truth table for $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \leftrightarrow r$	$(p \leftrightarrow q) \leftrightarrow r$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	F	T	F	F
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	T	F	F

Since all rows 8 (all possible cases for truth values of p, q, r) in the truth table for $(p \leftrightarrow q) \leftrightarrow r$ and $p \leftrightarrow (q \leftrightarrow r)$. have the same truth values, then these statement forms are logically equivalent. ■

2. Determine whether or not the statement $p \wedge q \to (p \to q)$ is a tautology or a contradiction.

Answer:

Truth table for $p \wedge q \to (p \to q)$

p	q	$p \wedge q$	$p \to q$	$p \wedge q \to (p \to q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Since all the rows (all possible cases for the truth values for p and q) in the truth table for $p \wedge q \to (p \to q)$ have the **true** truth value, then this statement form is a **tautology**. ■

3. Let p, q and r be statements such that $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ is **true**. Determine the truth value of $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$.

Answer: False

Truth table for $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \wedge (q \leftrightarrow r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	F	F
F	F	F	T	T	T

Notice from the above truth table that there are only 2 cases for p, q, r that make $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ true. These 2 cases are in the first and the last rows:
(i) p, q, r are all true or
(ii) p, q, r are all false.

That is, we only need to consider these two cases for $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$, which can be shown to be **false** for both cases (see the table below).

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \vee q \vee r$	$\sim p \vee \sim q \vee \sim r$	$(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$
T	T	T	F	F	F	T	F	F
F	F	F	T	T	T	F	T	F

Hence, when $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$ is **true**, the truth value of $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$ is always **false**. ■

4. Consider the following statement.

If I eat spicy food and I drink beer, then I feel sick or I have a bad dream.

(a) Write the **negation** of the above statement.

(b) Write the **contrapositive**, **inverse**, and **converse** of the above statement.

<u>Answer:</u>
Let p be "I eat spicy food ,"
q be "I drink beer,"
r be "I feel sick,"
s be "I have a bad dream."
Then the statement can be written as

$$(p \wedge q) \rightarrow (r \vee s).$$

(a) The **negation :**

$$\sim [(p \wedge q) \rightarrow (r \vee s)] \equiv (p \wedge q) \wedge \sim (r \vee s) \equiv (p \wedge q) \wedge (\sim r \wedge \sim s)$$

"'I eat spicy food and I drink beer, but(and) I do not feel sick and I do not have a bad dream."

(b) **Contrapositive:**

$$\sim (r \vee s) \rightarrow \sim (p \wedge q) \equiv (\sim r \wedge \sim s) \rightarrow (\sim p \vee \sim q)$$

"If I do not feel sick **and** *I do not have a bad dream, then I do not eat spicy food* **or** *I do not drink beer."*

Inverse:

$$\sim (p \wedge q) \rightarrow \sim (r \vee s) \equiv (\sim p \vee \sim q) \rightarrow (\sim r \wedge \sim s)$$

"If I do not eat spicy food **or** *I do not drink beer, then I do not feel sick* **and** *I do not have a bad dream."*

Converse

$$(r \vee s) \rightarrow (p \wedge q)$$

"If I feel sick or I have a bad dream, then I eat spicy food and I drink beer."

■

5. Use truth tables to determine whether the argument forms are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

(a)
$p \to \sim q$
$\sim q \to p$
$\therefore\ p \vee \sim q$

(b)
$r \vee \sim q$
$p \to q$
$\sim r$
$\therefore\ \sim p$

(c)
If I answer the quiz correctly, then I understand the class material.
I understand the class material.
∴ I answer the quiz correctly.

Answer:

(a)
$p \to \sim q$
$\sim q \to p$
$\therefore\ p \vee \sim q$

Truth table:

			Premises		Conclusion	
p	q	$\sim q$	$p \to \sim q$	$\sim q \to p$	$p \vee \sim q$	
T	T	F	F	T		
T	F	T	T	T	T	←-- critical row
F	T	F	T	T	F	←-- critical row: This has true premises but false conclusion.
F	F	T	T	F		

Notice that the columns 4 and 5 consist of the truth values of premises and rows 2 and 3 are critical rows (premises are all true). Since the row 3 (which is a critical row) has true premises with false conclusion, then this argument is **invalid**. ■

(b)
$r \vee \sim q$
$p \to q$
$\sim r$
$\therefore\ \sim p$

Truth table:

				Premises		Conclusion	
p	q	r	$\sim q$	$r \vee \sim q$	$p \to q$	$\sim r$	$\sim p$	
T	T	T	F	T	T	F	F	
T	T	F	F	F	T	T	F	
T	F	T	T	T	F	F	F	
T	F	F	T	T	F	T	F	
F	T	T	F	T	T	F	T	
F	T	F	F	F	T	T	T	
F	F	T	T	T	T	F	T	
F	F	F	T	T	T	T	T	←-- critical row

Notice that the columns 5, 6 and 7 give the truth values of premises and only row 8 (the last row) is the critical row. Since this critical row has all true premises with true conclusion, then this argument is **valid**. ■

(c) Let p be "I answer the quiz correctly"; and q be "I understand the class material."

	If I answer the quiz correctly, then I understand the class material.		$p \to q$
	I understand the class material.	$\equiv$	q
$\therefore$	I answer the quiz correctly.	$\therefore$	p

Truth table:

		Premises		Conclusion	
p	q	$p \to q$	q	p	
T	T	T	T	T	←-- critical row
T	F	F	F		
F	T	T	T	F	←-- critical row: This has true premises but false conclusion.
F	F	T	F		

Notice that the columns 3 and 4 consist of the truth values of premises and rows 1,3 are the critical rows. Since there is a critical row (row 3) that has true premises with false conclusion, then this argument is **invalid**. ■

6. Consider the following premises.

(i) It is not sunny this afternoon and it is colder than yesterday.

(ii) We will go swimming only if it is sunny.

(iii) If we do not go swimming, then we will take a canoe trip.

(iv) If we take a canoe trip, then we will be home by sunset.

From the above premises (i)-(iv), does the conclusion that *we go home by sunset* make a valid argument? Explain your answer by using rules of inferences.

Answer:

Let p be the statement "It is sunny this afternoon,"
q the statement "It is colder than yesterday,"
r the statement "We will go swimming,"
s the statement "We will take a canoe trip," and t the statement "We will be home by sunset."
Then the argument consists of the following premises and conclusion.

$\sim p \wedge q$
$r \to p$
$\sim r \to s$
$s \to t$
$\therefore\ t$

We construct an argument to show that our premises lead to the desired conclusion as follows.

(1) Simplification : premise (i)
$\sim p \wedge q$
$\therefore\ \sim p$

(2) Modus tollens: using premise (ii) and (1)

$r \to p$ premise (ii)
$\sim p$ from (1)
$\therefore \sim r$

(3) Modus ponens: using premise (iii) and (2)

$\sim r \to s$ premise (iii)
$\sim r$ from (2)
$\therefore s$

(4) Modus ponens: using premise (iv) and (3)

$s \to t$ premise (iv)
s from (3)
$\therefore t$

Therefore, we can conclude that "we will be home by sunset," from rules of inference used in (1)-(4).

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p, q, r, s, and t , such a truth table would have 32 rows. ■