Note Title 8/26/2016

Last time → Exact DE

$$\frac{dy}{dt} + u(t).y(t) = w(t)$$

$$\underbrace{e^{\int u(t)\,dt}}_{\text{integrating factor}}\frac{dy}{dt} + \underbrace{e^{\int u(t)\,dt}\left[u(t)\,y(t) - w(t)\right]}_{\boxed{f(y,t) = \frac{\partial F}{\partial t}}} = 0$$

$$M = g(y,t) = \frac{\partial F}{\partial y}$$

A3

step(i) $F(y,t) = \int M\,dy + \psi(t)$ — (A1)

$$= \int e^{\int u(t)\,dt}\,dy + \psi(t)$$

$$= [e^{\int u(t)\,dt}.y + C_1] + \psi(t) \quad \text{— (A2)}$$

step (ii)

$$\frac{\partial F(y,t)}{\partial t}$$

$$\frac{\partial F}{\partial t} = y.\,u(t)\,e^{\int u(t)\,dt} + \psi'(t) \quad \text{— (A3)}$$

$$\underline{y.u.e^{\int u(t)\,dt}} + \psi'(t) = \underline{e^{\int u(t)\,dt}[u\,y} - w]$$

$$\psi'(t) = -w(t).\,e^{\int u(t)\,dt} \quad \text{— (A4)}$$

step(iii) Find $\psi(t)$

integrating both sides of (A4) w.r.t. t

$$\int \psi'(t)\,dt = -\int w(t)\, e^{\int u(t)\,dt}\,dt$$

$$\psi(t) + C_2 = -\int w(t)\, e^{\int u(t)\,dt}\,dt \quad -(A5)$$

step (iv)

sub (A5) → (A2)

$$\therefore \quad F(y,t) = y(t)\, e^{\int u(t)\,dt} + C_1 - \int w(t)\, e^{\int u(t)\,dt}\,dt - C_2$$

$$C_3 = y(t)\, e^{\int u(t)\,dt} - \int w(t)\, e^{\int u(t)\,dt}\,dt + (C_1 - C_2)$$

$$y(t)\, e^{\int u(t)\,dt} = [C_3 - (C_1 - C_2)] + \int w(t)\, e^{\int u(t)\,dt}\,dt$$

$$\therefore \quad \boxed{y(t) = e^{-\int u(t)\,dt}\left[A + \int w(t)\, e^{\int u(t)\,dt}\,dt\right]}$$

where $A = C_3 - (C_1 - C_2) \quad — (15)$

Ex Find the general sol^n of $\boxed{\frac{dy}{dt} + 2ty = t}$

$u(t) = 2t \longrightarrow \int u(t)\,dt = \int 2t\,dt$

$w(t) = t \qquad\qquad\qquad\qquad = t^2 + C_1$

$$y(t) = e^{-(t^2+c_1)}\left[A + \int t\, e^{(t^2+c_1)}\, dt\right]$$

$$= e^{-t^2} e^{-c_1}\left[A + e^{c_1}\int t e^{t^2} dt\right]$$

$$= e^{-t^2} e^{-c_1} A + e^{-t^2}\left(\frac{1}{2} e^{t^2} + C_2\right)$$

$$= e^{-t^2} e^{-c_1} A + \frac{1}{2} + C_2 e^{-t^2}$$

$y'(t) = ?$

$$y(t) = B e^{-t^2} + \frac{1}{2} \quad ; \quad B = Ae^{-c_1} + C_2$$

Ex Wealth Accumulation

$\tilde{w}(t)$ = the amount of savings in the account @ time t.

$\tilde{y}(t)$ = the deposit rate

$c(t)$ = the withdrawal rate

$r(t)$ = the continuous compounding interest rate

$$\frac{d\tilde{w}(t)}{dt} = r(t).\tilde{w}(t) + (\tilde{y}(t) - c(t))$$

$$y(t) = -u(t)\, y(t) + w(t)$$

$$\tilde{w}(t) = e^{-\int -r(t)\,dt}\left[A + \int (\tilde{y}(t) - c(t))\, e^{\int -r(t)\,dt}\, dt\right]$$

2.1.5 Separable Differential Equations

$$\frac{dy}{dt} = F(y,t)$$

$$\frac{dy}{dt} = f(t)\, g(y) \qquad \text{— (1)}$$

we say (1) is Separable DE

or this DE is separable.

Ex

$$\frac{dy}{dt} = -2ty^2 \rightarrow f(t) = -2t, \quad g(y) = y^2$$

$$\frac{dy}{dt} = (t^2+t)y^2 \rightarrow f(t) = t^2+t, \quad g(y) = y^2$$

$$\frac{dy}{dt} = y^2 + t^2 \qquad \text{Not separable}$$

$$\frac{dy}{dt} = yt + t^2 \qquad \text{————}$$

A general method for solving (1) ⇒ "4-steps method"

step (i) write (1) as $\frac{dy}{dt} = f(t) . g(y)$

step (ii) separate the function $\frac{1}{g(y)} dy = f(t)\, dt$

step (iii) integrating both sides $\int \frac{1}{g(y)} dy = \int f(t)\, dt$

step (iv) evaluating two integrals

and then solve for $y(t)$ explicitly (if possible)

Note If $g(y)$ has a zero value @ $y = a$

$[\, g(y) = 0$ @ $y = a \,]$

$y(t) = a$ is also a solution of (1)

Ex solve $\frac{dy}{dt} = -2ty^2$ $\quad -2t \left[\frac{1}{t^2+c} \right]^2$

step (i) specify $f(t) = 2t$, $g(y) = -y^2$ $\quad \frac{dy}{dt} = f(t) \cdot g(y)$

step (ii) $\frac{1}{g(y)} dy = f(t)\, dt$

$-\frac{1}{y^2} dy = 2t\, dt$

step (iii) $\displaystyle\int -\frac{1}{y^2}\,dy = \int 2t\,dt$

step (iv) $\displaystyle\frac{1}{y} + C_1 = t^2 + C_2$

$$\frac{1}{y} = t^2 + C \quad ; C = C_2 - C_1$$

$$y(t) = \frac{1}{t^2 + C}$$

Check $\displaystyle y'(t) = -\frac{2t}{(t^2+C)^2}$ ✓

$g(y) = -y^2$ can be zero if $y = 0$

$\therefore$ $y(t) = 0$ is the other soln.

Ex $\displaystyle\frac{dy}{dt} = \frac{t^3}{y^6+1}$ Find $y(t) = ?$

$$\frac{y^7}{7} + y + C_1 = \frac{t^4}{4} + C_2$$

$$\boxed{\frac{y^7}{7} + y = \frac{t^4}{4} + C} \qquad C = C_2 - C_1$$

$g(y) = y^6 + 1$

Ex Economic Growth.

$Y(t)$ = the national product @ time t

$K(t)$ = the capital stock ______

$L(t)$ = the number of workers ______

for $\forall t \geq 0$

(a) $Y = A K^{1-\alpha} L^{\alpha}$: A Cobb-Douglas Prodn fn

'A' = the productivity level

(b) $I = S$

$$\frac{dK}{dt} = sY$$

(c) $L = \underbrace{L_0}_{\text{initial labor force level}} e^{\lambda t}$ the labor force grows exponentially at the rate λ

Find the time path of $K(t)$ ✓✓✓

set up the DE that involves $\frac{dK}{dt}$

$$\frac{dK}{dt} = s.\underbrace{Y}$$

$$= s A K^{1-\alpha} L^{\alpha}$$

$$= s A k^{1-\alpha} (L_0 e^{\lambda t})^{\alpha}$$

$$\frac{dK}{dt} = s A L_0^{\alpha} K^{1-\alpha} e^{\alpha \lambda t}$$

$$\frac{dy}{dt} + u(t) y(t) = w(t)$$

Use "separable" DE.

<u>step i</u> specify $g(K) = K^{1-\alpha}$

$$f(t) = \underbrace{(s A L_0^{\alpha})}_{\text{constant}} e^{\alpha \lambda t}$$

<u>step ii</u> $\frac{1}{g(K)} dK = f(t)\, dt$

$$\frac{1}{K^{1-\alpha}} dK = (s A L_0^{\alpha}) e^{\alpha \lambda t} dt$$

<u>step ii</u> $\int \frac{1}{K^{1-\alpha}} dK = \int (s A L_0^{\alpha}) e^{\alpha \lambda t} dt$

<u>Note</u> $0 < \alpha < 1$

<u>step iii</u> evaluting integrals

$$\frac{K^{-(1-\alpha)+1}}{-(1-\alpha)+1} + C_1 = (s A L_0^{\alpha}) \frac{e^{\alpha \lambda t}}{\alpha \lambda} + C_2$$

$$\frac{K^{\alpha}}{\alpha} + C_1 = (s A L_0^{\alpha}) \frac{e^{\alpha \lambda t}}{\alpha \lambda} + C_2$$

step iv $\quad K^{\alpha} = (sAL_0^{\alpha}) \frac{e^{\alpha\lambda t}}{\lambda} + C$

$$C = \alpha(C_2 - C_1)$$

$$K(t) = \left[sAL_0^{\alpha} \frac{e^{\alpha\lambda t}}{\lambda} + C \right]^{\frac{1}{\alpha}}$$ General Sol^n

Assuming $K(t)$ @ $t=0$ $\quad K(0) = K_0 > 0$

Find the Definite sol^n

$t=0$

$$K(0) = \left[sAL_0^{\alpha} + C \right]^{\frac{1}{\alpha}}$$

$$K_0^{\alpha} = [sAL_0^{\alpha} + C]$$

$$C = K_0^{\alpha} - sAL_0^{\alpha}$$

↳ sub into the general sol^n

$$K(t) = \left[K_0^{\alpha} + \frac{sAL_0^{\alpha}}{\lambda}(e^{\alpha\lambda t} - 1) \right]^{\frac{1}{\alpha}}$$ ⇒ Definite sol^n.

2.1.6. Bernoulli's Equation

$$\frac{dy}{dt} + u(t)y(t) = w(t)\,[y(t)]^{r} \qquad -(1)$$

r is a fixed real number

$u(t)$, $w(t)$ are continuous fn of t

$r = 0 \Rightarrow y'(t) + u(t)y(t) = w(t).$

$r = 1 \Rightarrow y'(t) + u(t)y(t) = w(t)y(t)$

$$y'(t) = w(t)y(t) - u(t)y(t)$$

$$= [w(t) - u(t)]y(t)$$

$$\frac{y'(t)}{y(t)} = w(t) - u(t).$$

for $r \neq 1$ and we look for a soln with

$y(t) > 0 \ \forall t$ so the $[y(t)]^r$ is always defined.

Dividing both sides by y^r

$$y^{-r}y'(t) + u(t)\,y^{1-r} = w(t) \qquad \text{—— } (\star).$$

Transforming by $z(t) = [y(t)]^{1-r}$

$$\Rightarrow z'(t) = \frac{dz}{dt} = (1-r)\,y(t)^{-r}\frac{dy}{dt}$$

$$z'(t) = (1-r)\, y(t)^{-r}\, y'(t)$$

or $$\boxed{\frac{1}{(1-r)}\, z'(t) = y(t)^{-r}\, y'(t)}$$

$$\frac{1}{1-r}\, z'(t) + u(t)\, z(t) = w(t) \quad \text{— (2)}$$

Note $$z(t) = y(t)^{1-r}$$

$$y(t) = [z(t)]^{\frac{1}{1-r}}$$