

Chapter 7 Vector Spaces

7.1 Vector Spaces and Subspaces

Definition 7.1 A *vector space* V is a set with the following properties:

- a) If $\mathbf{x} \in V$, then for any scalar $c \in \mathbf{R}$, $c\mathbf{x} \in V$.
- b) If $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x} + \mathbf{y} \in V$.

A vector space is closed under scalar multiplication and addition. That is, any multiple of an element in the vector space and the sum of any pair of elements are still elements in the vector space.

The set \mathbf{R}^n satisfies both properties and therefore is a vector space, but \mathbf{R}_+^n is not. Since the definition of the vector space does not specifically restrict itself to sets of vector, the set of matrices $\mathbf{R}^{m \times n}$ is also a vector space. A vector space can be a set of any object, provided that it satisfies both of the two properties, and we do not even need to define the addition and scalar multiplication in the conventional sense. This is why the definition of vector space in other textbooks, for example **Simon & Blume** [1994], page 751, requires ten conditions. See also **Leon** [1994], page 114 for examples and page 115 for Theorem 3.1.1.

Problem Leon [1994], page 116, # 4, 6, page 127, #3, 6, 7.

4. Show that $\mathbf{R}^{m \times n}$ with the usual addition and scalar multiplication of matrices is a vector space.
6. Let \mathbf{P} be the set of all polynomials. Show that \mathbf{P} with the usual addition and scalar multiplication of functions forms a vector space.

Definition 7.2 A set \mathbf{S} is a *subspace* of a vector space V if $\mathbf{S} \subseteq V$ and \mathbf{S} is also a vector space.

Example Let \mathbf{S} be the set of all the homogeneous solutions of the homogeneous system of linear equations $\mathbf{Ax} = \mathbf{0}$. The set \mathbf{S} is a vector space. Why? Then we can say that \mathbf{S} is a subspace of \mathbf{R}^n .

Problem Show that the set $\{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{b} \neq \mathbf{0}\}$ is not a subspace of \mathbf{R}^n .

Problem Simon & Blume [1994], page 755, #27.1, 27.2, 27.3, 27.4.

1. Which of the following are subspaces of \mathbf{R}^2 ? Explain your answer.
 - a. $\{(x, y) \mid x = 0\}$,
 - b. $\{(x, y) \mid x = 1\}$,
 - c. $\{(x, y) \mid 3x - 4y = 0\}$,

- d. $\{(x, y) \mid x^2 = y^2\}$,
- e. $\{(0, 1)\}$,
- f. $\{(x, y) \mid x + y = 0, x - y = 0\}$.

2. Show that the set \mathbf{R}_+^2 is closed under addition but is not a subspace.
3. Show that the set $\mathbf{W} = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbf{R}^2, x_1 = 0 \text{ or } x_2 = 0\}$ is closed under multiplication but it is not a subspace.
4. Show that the set $\mathbf{V} = \{(a, b, b, c) \mid a, b, c \in \mathbf{R}\}$ is a subspace in \mathbf{R}^4 .

Problem Leon [1994], page 127, #3, 8.

3. Determine whether or not the following are subspaces of $\mathbf{R}^{n \times n}$.
 - a. The set of all $n \times n$ diagonal matrices.
 - b. The set of all $n \times n$ lower triangular matrices.
 - c. The set of all $n \times n$ matrices \mathbf{A} such that $a_{12} = 1$.
 - d. The set of all $n \times n$ matrices \mathbf{B} such that $b_{11} = 0$.
 - e. The set of all symmetric $n \times n$ matrices.
 - f. The set of all singular $n \times n$ matrices.
8. Let \mathbf{A} be a particular matrix in $\mathbf{R}^{n \times n}$. Determine whether the following are subspaces of $\mathbf{R}^{n \times n}$.
 - a. $\mathcal{S}_1 = \{\mathbf{B} \mid \mathbf{B} \in \mathbf{R}^{n \times n}, \mathbf{AB} = \mathbf{BA}\}$,
 - b. $\mathcal{S}_2 = \{\mathbf{B} \mid \mathbf{B} \in \mathbf{R}^{n \times n}, \mathbf{AB} \neq \mathbf{BA}\}$,
 - c. $\mathcal{S}_3 = \{\mathbf{B} \mid \mathbf{B} \in \mathbf{R}^{n \times n}, \mathbf{BA} = \mathbf{0}\}$.

2. Spanning Sets and Basis of Vector Space

Definition 7.3 A vector space $V \in \mathbf{R}^n$ is *spanned* or *generated* by vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in V$ if for any vector $\mathbf{b} \in V$, \mathbf{b} can be written as a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$. That is, for any $\mathbf{b} \in V$, there exists some coefficients c_1, c_2, \dots, c_k such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_k\mathbf{a}_k = \mathbf{b}$.

Let $\mathbf{a}_1, \mathbf{a}_2$ be vectors in \mathbf{R}^2 . If $\mathbf{a}_1, \mathbf{a}_2$ are linearly independent, then $\mathbf{a}_1, \mathbf{a}_2$ span the vector space \mathbf{R}^2 , because any point \mathbf{b} in \mathbf{R}^2 can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$. Why?

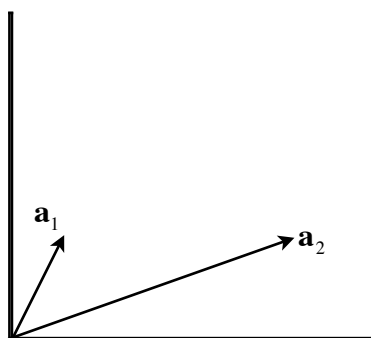


Figure 7.1 The vector \mathbf{a}_1 , \mathbf{a}_2 span the vector space \mathbf{R}^2 .

Problem Simon & Blume [1994], page 246, #11.9, 11.10.

9. a) Write $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.
- b. Write $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.
10. Do $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}$ span \mathbf{R}^3 ?

Problem Leon [1994], page 141, #15, 16.

15. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be vectors that span \mathcal{V} and let \mathbf{a} be any other vector in \mathcal{V} . Show that $\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are linearly dependent.
16. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be linearly independent vectors in a vector space \mathcal{V} . Show that $\mathbf{a}_2, \dots, \mathbf{a}_k$ cannot span \mathcal{V} .

Problem Show that any two vector space are equal if, and only if, they have the same spanning set.

Definition 7.4 Vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are a *basis* of the vector space $\mathcal{V} \in \mathbf{R}^n$, if the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ span the vector space \mathcal{V} and are linearly independent.

We can write a summarizing corollary as follows.

Corollary 7.1 Let \mathbf{A} be a square matrix in $\mathbf{R}^{n \times n}$. The following statements are equivalent.

1. The columns of \mathbf{A} are a basis of \mathbf{R}^n .
2. The rows of \mathbf{A} are a basis of \mathbf{R}^n .
3. The columns of \mathbf{A} are linearly independent.
4. The rows of \mathbf{A} are linearly independent.
5. \mathbf{A}^{-1} exists.

6. $|\mathbf{A}| \neq 0$, i.e., \mathbf{A} is nonsingular.
7. $\text{rank } \mathbf{A} = \text{rank } \mathbf{A}^T = n$.
8. $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any given vector $\mathbf{b} \in \mathbf{R}^n$.

Proof Exercise. \square

Example The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in Figure 7.2 below span the vector space \mathbf{R}^2 but they are not a basis. However, any pair of vectors out of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is a basis of \mathbf{R}^2 .

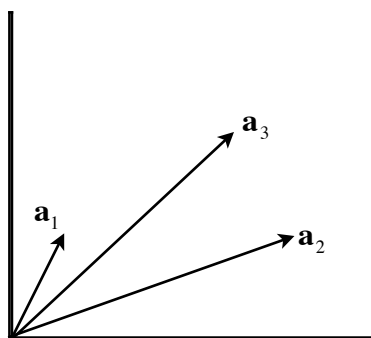


Figure 7.2 The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ span \mathbf{R}^2 , but they are not a basis of \mathbf{R}^2 .

Problem Simon & Blume [1994], page 249, #11.12, 11.14.

12. Which of the following are bases of \mathbf{R}^2 ?

- a. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}$
- b. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}$
- c. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}$
- d. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

14. Which of the following are bases in \mathbf{R}^3 ?

- a. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- b. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
- c. $\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$

$$\text{d. } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be a basis of a vector space V . Show that any vector $\mathbf{a} \in V$ can be written as a unique linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$.

It is obvious that for a given vector space, there can be more than one set of vectors that is the basis. For example, as demonstrated in Figure 7.2 any two linearly independent vectors in \mathbf{R}^2 is a basis of \mathbf{R}^2 . We can show in general that for a given vector space V that has $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ as a basis, any other k linearly independent vectors in V are also a basis of V .

Theorem 7.1 Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be a basis of a vector space V in \mathbf{R}^n . Suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are any k linearly independent vectors in V . Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are also a basis of V .

Proof The proof is divided into three parts:

- Show that there exists a vector $\mathbf{x}_i \in \mathbf{R}^k$ such that $\mathbf{v}_i = \mathbf{A}\mathbf{x}_i$, for any $i = 1, 2, \dots, k$, where $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_k]$.
- Show that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are linearly independent and thus the square matrix $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_k] \in \mathbf{R}^{k \times k}$ are nonsingular.
- Show that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ span V .

(a) Since $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are a basis of V , for any $\mathbf{v}_i \in V$, it can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$. That is, for any $\mathbf{v}_i \in V$, there exist scalars $x_{1i}, x_{2i}, \dots, x_{ki}$ such that

$$\begin{aligned} \mathbf{v}_i &= x_{1i}\mathbf{a}_1 + x_{2i}\mathbf{a}_2 + \dots + x_{ki}\mathbf{a}_k \\ &= [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_k] \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ki} \end{bmatrix} \\ &= \mathbf{A}\mathbf{x}_i, \end{aligned}$$

with $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_k]$ and $\mathbf{x}_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ki} \end{bmatrix} \in \mathbf{R}^k, i = 1, 2, \dots, k.$

(b) Let there be a linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$. We have to show that such a linear combination must be trivial, i.e., $c_1 = c_2 = \cdots = c_k = 0$. From (a), we have

$$\begin{aligned} \mathbf{0} &= \mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_k\mathbf{A}\mathbf{x}_k \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k. \end{aligned}$$

Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent, we can conclude that $c_1 = c_2 = \cdots = c_k = 0$.

(c) From part (a), we can write

$$\begin{aligned} [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k] &= \mathbf{A}[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_k] \\ \mathbf{V} &= \mathbf{A}\mathbf{X} \\ \mathbf{A} &= \mathbf{V}\mathbf{X}^{-1}, \end{aligned}$$

where $\mathbf{X} \in \mathbf{R}^{k \times k}$. The last equality follows from Theorem 5.6 that if the columns of the square matrix \mathbf{X} are linearly independent, the matrix \mathbf{X} must be invertible. We need to show that for any $\mathbf{b} \in V$, \mathbf{b} can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Since $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ span V , we can write

$$\begin{aligned} \mathbf{b} &= d_1\mathbf{a}_1 + d_2\mathbf{a}_2 + \cdots + d_k\mathbf{a}_k \\ &= \mathbf{A}\mathbf{d} \\ &= \mathbf{V}\mathbf{X}^{-1}\mathbf{d} \\ &= \mathbf{V}\mathbf{y}. \end{aligned}$$

That is, the vector \mathbf{b} can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. \square

Theorem 7.1 implies that for a given vector space there could be many possible bases. However, we can show that of all the possible bases, they will have one thing in common: all bases of a vector space consist of the same number of vectors. This fact follows directly from the next theorem.

Theorem 7.2 Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be a basis of a vector space $V \in \mathbf{R}^n$. Any set of more than k vectors in V is linearly dependent.

Proof By Lemma 6.1, it suffices to show that any set of $k+1$ vectors must be linearly dependent. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}$ be any arbitrary $k+1$ vectors in V . Since each vector $\mathbf{v}_i \in V$, it can be written as a linear combination of the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$,

$$\mathbf{v}_i = \mathbf{A}\mathbf{x}_i,$$

where $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_k]$ and for some vector of coefficients $\mathbf{x}_i \in \mathbf{R}^k$, $i = 1, 2, \dots, k+1$.

To show that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}$ are linearly dependent, we need to show that there are coefficients b_1, b_2, \dots, b_{k+1} , at least one of which is nonzero, such that $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_{k+1}\mathbf{v}_{k+1} = \mathbf{0}$. Using the fact that $\mathbf{v}_i = \mathbf{A}\mathbf{x}_i$,

$$\begin{aligned} \mathbf{0} &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_{k+1}\mathbf{v}_{k+1} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_{k+1}]\mathbf{b} \\ &= [\mathbf{A}\mathbf{x}_1 \ \mathbf{A}\mathbf{x}_2 \ \dots \ \mathbf{A}\mathbf{x}_{k+1}]\mathbf{b} \\ &= \mathbf{A}[\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_{k+1}]\mathbf{b} \\ &= \mathbf{A}\mathbf{y}. \end{aligned}$$

where $\mathbf{y} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_{k+1}]\mathbf{b}$.

This means that a linear combination of the columns of \mathbf{A} , which are the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$, is a zero vector. Since the basis is by definition linearly independent, the only possibility we have $\mathbf{y} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_{k+1}]\mathbf{b} = \mathbf{0}$. By Theorem 6.4, the $k+1$ vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k+1}$ in \mathbf{R}^k are linearly dependent and we can find a nonzero vector \mathbf{b} such that $[\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_{k+1}]\mathbf{b} = \mathbf{0}$ and this completes the proof. \square

With this theorem, it follows directly that every basis of a vector space must have the same number of elements.

Corollary 7.2 Every basis of a vector space $V \in \mathbf{R}^n$ has equal number of elements.

Proof Exercise. \square

Problem Suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent in a subspace in V but cannot span V . Show that there exists another vector \mathbf{v}_{k+1} in V such that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent.

3. Dimension of Vector Space

We can then define the dimension of a vector space to be this number of vectors in any basis of the vector space.

Definition 7.5 The *dimension* of the vector space V , denoted $\dim(V)$, is the number of the vectors in its basis.

Example Let $\mathbf{a}_1, \mathbf{a}_2$ be linearly independent vectors in \mathbf{R}^3 . The vectors $\mathbf{a}_1, \mathbf{a}_2$ cannot span \mathbf{R}^3 . But $\mathbf{a}_1, \mathbf{a}_2$ can span

$$L(\mathbf{a}_1, \mathbf{a}_2) = \left\{ \mathbf{x} \in \mathbf{R}^3 \mid \mathbf{x} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2, c_1, c_2 \in \mathbf{R} \right\}.$$

Since $\mathbf{a}_1, \mathbf{a}_2$ are linearly independent, they are a basis for $L(\mathbf{a}_1, \mathbf{a}_2)$, and the dimension of $L(\mathbf{a}_1, \mathbf{a}_2)$ is 2.

Theorem 7.3 The dimension of \mathbf{R}^n is n . That is, every basis of \mathbf{R}^n has n members.

Proof Exercise. \square

Problem Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be linearly independent vectors in a vector space V . Show that the dimension of V is at least k .

Problem Let V be a subspace of W . Show that

- $\dim(V) \leq \dim(W)$, and
- $\dim(V) < \dim(W)$ if $V \neq W$.

Solution a) $\dim(V) \leq \dim(W)$. Suppose toward contradiction that $\dim(V) > \dim(W)$. Specifically, there exist linearly independent vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k$ that form a basis of W , and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+h}$ that form the basis of V , for some positive integer h . Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+h}$ are a basis of V , they are linearly independent by definition. By Lemma 6.1, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are also linearly independent and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V \subseteq W$. By Theorem 7.1, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are also a basis of W and \mathbf{v}_{k+1} can thus be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. This means $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+h}$ are linearly dependent and this is a contradiction.

b) $\dim(V) < \dim(W)$ if $V \neq W$. Assume that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are a basis of V . Since $V \subseteq W$ and $V \neq W$, there exists $\mathbf{w} \in W$ such that $\mathbf{w} \notin V$. By the

definition of vector space, the vector \mathbf{w} cannot be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, and thus $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{w}$ are vectors in \mathcal{W} that are linearly independent. Thus, to span \mathcal{W} , we need at least $k+1$ linearly independent vectors and the dimension of \mathcal{W} is at least $k+1 > \dim(\mathcal{V})$. \square

Problem Fraleigh & Beauregard [1995], page 203, #34. Let \mathcal{W} and \mathcal{U} be subspaces of a vector space \mathcal{V} , and let $\mathcal{W} \cap \mathcal{U} = \{\mathbf{0}\}$. Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ be a basis of \mathcal{W} and $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a basis of \mathcal{U} . Prove that, if each vector \mathbf{v} in \mathcal{V} is expressible in the form $\mathbf{v} = \mathbf{w} + \mathbf{u}$ for some $\mathbf{w} \in \mathcal{W}$ and some $\mathbf{u} \in \mathcal{U}$, then $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is a basis of \mathcal{V} .

7.4 Row, Column and Null Spaces

Associated with each matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, we can define at least three subspaces as follows:

Definition 7.6 Let \mathbf{A} be an $m \times n$ matrix. Define

$$\begin{aligned} \text{Row}(\mathbf{A}) &= \textit{Row space} \text{ of } \mathbf{A} \\ &= \{ \mathbf{c} \mid \mathbf{c} \text{ is a linear combination of rows of } \mathbf{A} \} \\ &= \{ \mathbf{c} \mid \mathbf{A}^T \mathbf{y} = \mathbf{c}, \mathbf{y} \in \mathbf{R}^m \} \\ \text{Col}(\mathbf{A}) &= \textit{Column space} \text{ of } \mathbf{A} \\ &= \{ \mathbf{b} \mid \mathbf{b} \text{ is a linear combination of columns of } \mathbf{A} \} \\ &= \{ \mathbf{b} \mid \mathbf{A} \mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbf{R}^n \} \\ \text{Null}(\mathbf{A}) &= \textit{Null space} \text{ of } \mathbf{A} \\ &= \{ \mathbf{x} \mid \mathbf{x} \text{ is a solution of the homogeneous system } \mathbf{A} \mathbf{x} = \mathbf{0} \} \\ &= \{ \mathbf{x} \mid \mathbf{A} \mathbf{x} = \mathbf{0}, \mathbf{x} \in \mathbf{R}^n \}. \end{aligned}$$

Problem Show that $\text{Col}(\mathbf{A}^T) = \text{Row}(\mathbf{A})$.

Problem Show that $\text{Row}(\mathbf{A})$ and $\text{Null}(\mathbf{A})$ are subspaces of \mathbf{R}^n , and $\text{Col}(\mathbf{A})$ is a subspace of \mathbf{R}^m .

Problem Simon & Blume [1994], page 759, #27.7 For each of the following matrices compute bases for the row column and null spaces.

- a. $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$,
- b. $\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 5 \end{bmatrix}$,
- c. $\begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$,
- d. $\begin{bmatrix} 4 & 1 & -5 & 1 \\ 8 & 5 & -10 & 8 \\ -4 & 2 & 7 & 5 \end{bmatrix}$.

Problem Leon [1994], page 166, #6. How many solutions will the linear system $\mathbf{Ax} = \mathbf{b}$ have if \mathbf{b} is in the column space of \mathbf{A} and the column vectors of \mathbf{A} are linearly dependent?

5. Dimension of Row Space and Rank

The main result of this chapter is to show that $\text{Row}(\mathbf{A})$ and $\text{Col}(\mathbf{A})$ have the same dimension, which turns out to be equal to $\text{rank } \mathbf{A}$. We will also show how the dimension of $\text{Null}(\mathbf{A})$ relates to that of $\text{Col}(\mathbf{A})$. First we need the following results.

Theorem 7.4 The elementary row operations on \mathbf{A} do not change its row space.

Proof There are three types of elementary row operations: multiplying a row by a nonzero scalar, interchanging the positions of any pair of rows and adding to a row a multiple of another row. It is obvious that the theorem is true for the first two types. For the last type, we can assume without loss of generality that the second row is multiplied by a constant and then added to the first row, and then show that the new set of rows still spans the same subspace $\text{Row}(\mathbf{A})$.

Let rows of \mathbf{A} be denoted by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ and after the elementary row operation they become $(\mathbf{a}_1 + c\mathbf{a}_2), \mathbf{a}_2, \dots, \mathbf{a}_m$, $c \neq 0$. We will show that for any element $\mathbf{d} \in \text{Row}(\mathbf{A})$, \mathbf{d} can be written as a linear combination of $(\mathbf{a}_1 + c\mathbf{a}_2), \mathbf{a}_2, \dots, \mathbf{a}_m$. Since $\mathbf{d} \in \text{Row}(\mathbf{A})$, there are some constants x_1, x_2, \dots, x_m such that

$$\mathbf{d} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_m\mathbf{a}_m.$$

The question is whether we can find some constants z_1, z_2, \dots, z_m such that

$$\begin{aligned} \mathbf{d} &= x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_m \mathbf{a}_m \\ &= z_1 (\mathbf{a}_1 + c \mathbf{a}_2) + z_2 \mathbf{a}_2 + \cdots + z_m \mathbf{a}_m. \end{aligned}$$

The last term can be rearranged as $z_1 \mathbf{a}_1 + (cz_1 + z_2) \mathbf{a}_2 + \cdots + z_m \mathbf{a}_m$, which is equal to \mathbf{d} if we set all $z_i = x_i$ except $cz_1 + z_2 = x_2$, which means $z_2 = x_2 - cx_1$. The reverse is as simple and is left as an exercise. \square

We now have the following corollary.

Corollary 7.3 For any matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\text{rank } \mathbf{A} = \dim (\text{Row}(\mathbf{A}))$.

Proof Let \mathbf{U} be the row echelon matrix resulting from a series of elementary row operations on \mathbf{A} . Since the nonzero rows of the echelon matrix \mathbf{U} are linearly independent (why?), the nonzero rows of \mathbf{U} form a basis of $\text{Row}(\mathbf{U})$ and we have $\text{rank } \mathbf{U} = \dim (\text{Row}(\mathbf{U}))$. Since we know from Corollary 6.2 that elementary row operation does not change the number of linearly independent rows, and Theorem 6.2 that the rank of a matrix is equal to the number of linearly independent rows, we have

$$\begin{aligned} \text{rank } \mathbf{A} &= \text{rank } \mathbf{U} \\ &= \dim (\text{Row}(\mathbf{U})) \\ &= \dim (\text{Row}(\mathbf{A})). \end{aligned}$$

The last equality follows from Theorem 7.4 that $\text{Row}(\mathbf{A}) = \text{Row}(\mathbf{U})$. \square

1. Dimensions of Row and Column Spaces

We will show next that the dimension of the row space and column space of any matrix are equal. The following follows **Leon**, [1994], Theorem 3.6.5, page 163.

Theorem 7.5 For any matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\dim (\text{Row}(\mathbf{A})) = \dim (\text{Col}(\mathbf{A}))$.

Proof Let \mathbf{A} be a matrix of rank r , $r \leq \min\{m, n\}$, whose row echelon form is \mathbf{U} with r nonzero rows. The matrix \mathbf{U} also contains r columns with pivot elements. Delete the other $n - r$ columns of \mathbf{U} and corresponding $n - r$ columns of \mathbf{A} , and call the resulting matrices \mathbf{U}_L and \mathbf{A}_L respectively.

We know that the columns of \mathbf{U}_L are linearly independent (why?) and the homogeneous system of linear equations $\mathbf{U}_L \mathbf{x} = \mathbf{0}$ is equivalent to $\mathbf{A}_L \mathbf{x} = \mathbf{0}$ in the

sense that any solution \mathbf{x} of $\mathbf{U}_L \mathbf{x} = \mathbf{0}$ is also a solution of $\mathbf{A}_L \mathbf{x} = \mathbf{0}$. Since the columns of \mathbf{U}_L are linearly independent, $\mathbf{x} = \mathbf{0}$ is the unique solution of $\mathbf{U}_L \mathbf{x} = \mathbf{0}$ and $\mathbf{A}_L \mathbf{x} = \mathbf{0}$. By definition, the columns of \mathbf{A}_L are linearly independent. Therefore there are at least r columns of \mathbf{A} that are linearly independent. In other words, $\dim(\text{Col}(\mathbf{A})) \geq r = \dim(\text{Row}(\mathbf{A}))$. To show the reverse of the inequality, we write

$$\begin{aligned} \dim(\text{Row}(\mathbf{A})) &= \dim(\text{Col}(\mathbf{A}^T)) \\ &\geq \dim(\text{Row}(\mathbf{A}^T)) \\ &= \dim(\text{Col}(\mathbf{A})). \end{aligned}$$

This proves the theorem. \square

Corollary 7.4 For any matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\text{rank } \mathbf{A}^T = \text{rank } \mathbf{A}$.

Proof Immediate from Corollary 7.3 and Theorem 7.5. \square

Note that this Corollary 7.4 is stronger than Corollary 4.3 that the matrix \mathbf{A} needs not be square and nonsingular.

Problem Leon [1994], page 166, #9. Let \mathbf{A} and \mathbf{B} be row equivalent matrices.

- Show that the dimension of the column space of \mathbf{A} equals the dimension of the column space of \mathbf{B} .
- Are the column spaces of the two matrices necessarily the same? Justify your answer.

A summarizing corollary for a square matrix \mathbf{A} can be written as,

Corollary 7.5 Let \mathbf{A} be a matrix in $\mathbf{R}^{n \times n}$. The following statements are equivalent.

- $\text{Row}(\mathbf{A}) = \text{Col}(\mathbf{A}) = \mathbf{R}^n$.
- $\dim(\text{Row}(\mathbf{A})) = \dim(\text{Col}(\mathbf{A})) = n$.
- The columns of \mathbf{A} are a basis of \mathbf{R}^n .
- The rows of \mathbf{A} are a basis of \mathbf{R}^n .
- The columns of \mathbf{A} are linearly independent.
- The rows of \mathbf{A} are linearly independent.
- \mathbf{A}^{-1} exists.
- $|\mathbf{A}| \neq 0$, i.e., \mathbf{A} is nonsingular.
- $\text{rank } \mathbf{A} = \text{rank } \mathbf{A}^T = n$.
- $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution for any given vector $\mathbf{b} \in \mathbf{R}^n$.

Proof Exercise. \square

7. Linearly Independent Rows and Row Space

For any matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, we can state the following corollary by using Theorem 6.3 that states that the rank of matrix \mathbf{A} is equal to the number of linearly independent rows.

Corollary 7.6 For any matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, we have

$$\begin{aligned} \text{rank } \mathbf{A} &= \text{rank } \mathbf{A}^T \\ &= \dim(\text{Row}(\mathbf{A})) = \dim(\text{Col}(\mathbf{A})) \\ &= \text{the number of linearly independent rows of } \mathbf{A} \\ &= \text{the number of linearly independent columns of } \mathbf{A}. \end{aligned}$$

Proof Exercise. \square

By definition, the rows of matrix \mathbf{A} span $\text{Row}(\mathbf{A})$ and the columns of \mathbf{A} span $\text{Col}(\mathbf{A})$. We can show in the next theorem that the set of all the linearly independent rows of \mathbf{A} is a basis of $\text{Row}(\mathbf{A})$, and by the same reasoning the set of all the linearly independent columns of \mathbf{A} is a basis of $\text{Col}(\mathbf{A})$.

Theorem 7.6 Suppose that the rows of a matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$ can be separated such that the rows $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are linearly independent and that each of the rows $\mathbf{a}_{k+1}, \mathbf{a}_{k+2}, \dots, \mathbf{a}_m$ can be written as a linear combination of rows $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$. Then the rows $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are a basis of $\text{Row}(\mathbf{A})$.

Proof Exercise. \square

1. Fundamental Theorem of Linear Algebra

The last major result of this chapter is so important that it is called the Fundamental Theorem of Linear Algebra. It shows the relationship between the dimension of the row space and the null space. This proof follows **Simon & Blume** [1994], Theorem 27.10, page 777.

Theorem 7.7 For any matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\dim(\text{Null}(\mathbf{A})) = n - \text{rank } \mathbf{A}$.

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ form a basis of the subspace $\text{Null}(\mathbf{A})$. If $k = n$, then $\text{Null}(\mathbf{A}) = \mathbf{R}^n$, and it follows that $\mathbf{A} = \mathbf{0}$. (Exercise) Then $\text{rank } \mathbf{A} = 0$ and the theorem holds.

If $k < n$, we will show that the column space of \mathbf{A} has dimension $n - k$. We first argue that there exists $\mathbf{u}_{k+1}, \mathbf{u}_{k+2}, \dots, \mathbf{u}_n$ such that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ form a basis of \mathbf{R}^n (See **Simon & Blume** [1994], Lemma 27.4, page 776). Then we will show that the vectors $\mathbf{A}\mathbf{u}_{k+1}, \mathbf{A}\mathbf{u}_{k+2}, \dots, \mathbf{A}\mathbf{u}_n$ form a basis of $\text{Col}(\mathbf{A})$ and thus $\text{Col}(\mathbf{A})$ has dimension $n - k$.

First we will show that $\mathbf{A}\mathbf{u}_{k+1}, \mathbf{A}\mathbf{u}_{k+2}, \dots, \mathbf{A}\mathbf{u}_n$ are linearly independent by showing that any linear combination of $\mathbf{A}\mathbf{u}_{k+1}, \mathbf{A}\mathbf{u}_{k+2}, \dots, \mathbf{A}\mathbf{u}_n$ that is zero has to be trivial. That is,

$$\begin{aligned} \mathbf{0} &= c_{k+1}\mathbf{A}\mathbf{u}_{k+1} + c_{k+2}\mathbf{A}\mathbf{u}_{k+2} + \dots + c_n\mathbf{A}\mathbf{u}_n \\ &= \mathbf{A}(c_{k+1}\mathbf{u}_{k+1} + c_{k+2}\mathbf{u}_{k+2} + \dots + c_n\mathbf{u}_n). \end{aligned}$$

which means $c_{k+1}\mathbf{u}_{k+1} + c_{k+2}\mathbf{u}_{k+2} + \dots + c_n\mathbf{u}_n$ is in $\text{Null}(\mathbf{A})$ and by definition can be written as a linear combination of the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$, so

$$\begin{aligned} c_{k+1}\mathbf{u}_{k+1} + c_{k+2}\mathbf{u}_{k+2} + \dots + c_n\mathbf{u}_n &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_k\mathbf{u}_k \\ \mathbf{0} &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_k\mathbf{u}_k - c_{k+1}\mathbf{u}_{k+1} - c_{k+2}\mathbf{u}_{k+2} - \dots - c_n\mathbf{u}_n. \end{aligned}$$

Since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are linearly independent, $c_1 = c_2 = \dots = c_k = c_{k+1} = \dots = c_n = 0$ and thus $\mathbf{A}\mathbf{u}_{k+1}, \mathbf{A}\mathbf{u}_{k+2}, \dots, \mathbf{A}\mathbf{u}_n$ are linearly independent.

Finally, we will show that $\mathbf{A}\mathbf{u}_{k+1}, \mathbf{A}\mathbf{u}_{k+2}, \dots, \mathbf{A}\mathbf{u}_n$ span the subspace $\text{Col}(\mathbf{A})$. Let $\mathbf{b} \in \text{Col}(\mathbf{A})$, so there exists $\mathbf{x} \in \mathbf{R}^n$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$. Since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ span \mathbf{R}^n ,

$$\begin{aligned} \mathbf{b} &= \mathbf{A}\mathbf{x} = \mathbf{A}(d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \dots + d_n\mathbf{u}_n) \\ &= d_1\mathbf{A}\mathbf{u}_1 + d_2\mathbf{A}\mathbf{u}_2 + \dots + d_k\mathbf{A}\mathbf{u}_k + d_{k+1}\mathbf{A}\mathbf{u}_{k+1} + \dots + d_n\mathbf{A}\mathbf{u}_n \\ &= d_{k+1}\mathbf{A}\mathbf{u}_{k+1} + \dots + d_n\mathbf{A}\mathbf{u}_n. \end{aligned}$$

The last equation is due to the fact that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are in $\text{Null}(\mathbf{A})$. The subspace $\text{Col}(\mathbf{A})$ is then spanned by $n - k$ linearly independent vectors and so its dimension is also $n - k$ as needed. \square

Problem Johnson, Riess & Arnold [1998], page 164, #39, 40.

39. Prove that a square matrix \mathbf{A} is nonsingular if and only if the dimension of $\text{Null}(\mathbf{A})$ is zero.
40. Let \mathbf{A} be an $m \times m$ nonsingular matrix and \mathbf{B} an $m \times m$ matrix. Prove that $\text{Null}(\mathbf{AB}) = \text{Null}(\mathbf{B})$ and conclude that $\text{rank } \mathbf{AB} = \text{rank } \mathbf{B}$.

Problem Simon & Blume [1994], page 771, #27.15, 27.16.

15. Show the following:
- $\text{Col}(\mathbf{AB})$ is a subspace of $\text{Col}(\mathbf{A})$ whenever \mathbf{AB} is well-defined.
 - If \mathbf{B} is square and nonsingular, then $\text{Col}(\mathbf{AB}) = \text{Col}(\mathbf{A})$.
- 27.16 Show the following:
- $\text{Null}(\mathbf{B})$ is a subspace of $\text{Null}(\mathbf{AB})$ whenever \mathbf{AB} is well-defined.
 - If \mathbf{A} is square and nonsingular, then $\text{Null}(\mathbf{AB}) = \text{Null}(\mathbf{B})$.

Problem Leon [1994], page 166, #7, 13-16.

7. Let \mathbf{A} be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbf{R}^m$ and suppose that $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$.
- What can you conclude about the column vectors of \mathbf{A} ? Are they linearly independent? Do they span \mathbf{R}^m ? Explain.
 - How many solutions will the system $\mathbf{Ax} = \mathbf{b}$ have if \mathbf{b} is not in the column space of \mathbf{A} ? How many solutions will there be if \mathbf{b} is in the column space of \mathbf{A} ? Explain.
13. Suppose that \mathbf{A} and \mathbf{B} are $n \times n$ matrices with the property that $\mathbf{Ax} = \mathbf{Bx}$ for all $\mathbf{x} \in \mathbf{R}^n$. Show that
- $\text{Null}(\mathbf{A} - \mathbf{B}) = \mathbf{R}^n$.
 - $\mathbf{A} - \mathbf{B}$ must have rank 0 and consequently $\mathbf{A} = \mathbf{B}$.
14. Let \mathbf{A} and \mathbf{B} be $n \times n$ matrices. Show that $\mathbf{AB} = \mathbf{0}$ if and only if the column space of \mathbf{B} is a subspace of the nullspace of \mathbf{A} .
15. Let $\mathbf{A} \in \mathbf{R}^{m \times n}$, $\mathbf{b} \in \mathbf{R}^m$ and let \mathbf{x}_0 be a particular solution to the system $\mathbf{Ax} = \mathbf{b}$. Prove the following.
- A vector \mathbf{y} in \mathbf{R}^n will be a solution to $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, where $\mathbf{z} \in \text{Null}(\mathbf{A})$.
 - If $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$, then the solution \mathbf{x}_0 is unique.
16. Let \mathbf{x} and \mathbf{y} be nonzero vectors in \mathbf{R}^m and \mathbf{R}^n , respectively, and let $\mathbf{A} = \mathbf{xy}^T$.
- Show that $\{\mathbf{x}\}$ is a basis for the column space of \mathbf{A} and that $\{\mathbf{y}\}$ is a basis for the row space of \mathbf{A} .
 - What is the dimension of $\text{Null}(\mathbf{A})$?

7.9 Orthogonality between Row and Null Spaces

The next theorem shows that the $\text{Row}(\mathbf{A})$ and $\text{Null}(\mathbf{A})$ are orthogonal.

Theorem 7.8 Let \mathbf{A} be a matrix of order $m \times n$. We know that both $\text{Row}(\mathbf{A})$ and $\text{Null}(\mathbf{A})$ are subspaces of \mathbf{R}^n . We have

- If $\mathbf{c}_0 \in \text{Row}(\mathbf{A})$ and $\mathbf{x}_0 \in \text{Null}(\mathbf{A})$, then $\mathbf{c}_0^T \mathbf{x}_0 = 0$.
- The only common vector in $\text{Row}(\mathbf{A})$ and $\text{Null}(\mathbf{A})$ is the zero vector. That is, a vector $\mathbf{c}_0 \in \text{Row}(\mathbf{A})$ and $\mathbf{c}_0 \in \text{Null}(\mathbf{A})$ if, and only if, $\mathbf{c}_0 = \mathbf{0}$.

Proof a) Since $\mathbf{c}_0 \in \text{Row}(\mathbf{A})$, there exists $\mathbf{y} \in \mathbf{R}^m$ such that $\mathbf{c}_0 = \mathbf{A}^T \mathbf{y}$. Since $\mathbf{x}_0 \in \text{Null}(\mathbf{A})$, we know that $\mathbf{A} \mathbf{x}_0 = \mathbf{0}$. Then, $\mathbf{c}_0^T \mathbf{x}_0 = (\mathbf{A}^T \mathbf{y})^T \mathbf{x}_0 = \mathbf{y}^T \mathbf{A} \mathbf{x}_0 = \mathbf{y}^T \mathbf{0} = 0$.

b) From (a), if $\mathbf{c}_0 \in \text{Row}(\mathbf{A})$ and $\mathbf{c}_0 \in \text{Null}(\mathbf{A})$, we have $\mathbf{c}_0^T \mathbf{c}_0 = 0$. So $\mathbf{c}_0 = \mathbf{0}$ because the inner product of a vector is zero if, and only if, it is a zero vector. Now, if $\mathbf{c}_0 = \mathbf{0}$, $\mathbf{c}_0 \in \text{Row}(\mathbf{A})$ because $\mathbf{A}^T \mathbf{0} = \mathbf{c}_0 = \mathbf{0}$, and $\mathbf{c}_0 \in \text{Null}(\mathbf{A})$ because $\mathbf{A} \mathbf{c}_0 = \mathbf{A} \mathbf{0} = \mathbf{0}$. \square