Note Title 8/19/2016

(6) The Domar Growth Model

Framework : $\Delta I(t) \longrightarrow$ The AD effect

$\longrightarrow$ The productive capacity of the economy.

(1) The AD effect

$$\underbrace{\begin{aligned} Y &= C + I \\ Y - C &= I \\ S &= I \\ dS &= dI \end{aligned} \quad \Bigg| \quad \begin{aligned} C &= a + bY \quad ; \; b = MPC \\ S &= Y - C \\ S &= -a + (1-b)Y \\ dS &= (1-b)\, dY \end{aligned}}$$

$\therefore \; (1-b) = MPS$

$$MPS \, dY = dI$$

$$dY = \frac{1}{s} dI \quad ; \; MPS = s$$

$$\text{or} \quad dY = k\, dI \quad ; \; k = \frac{1}{s}$$

$$\boxed{\frac{dY}{dt} = k \frac{dI}{dt}} \quad \text{— (1)}$$

(2) The Productive Capacity effect

ρ = output per capital stock

= constant

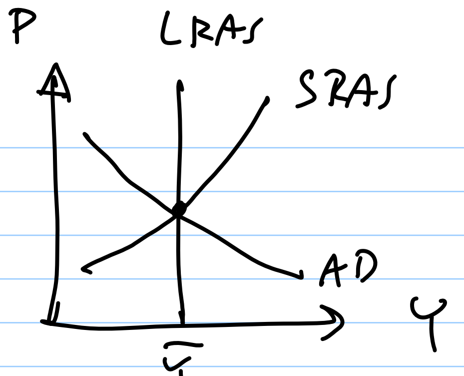
$$\rho = \frac{\kappa}{K(t)}$$

κ = potential / capacity output flow per year.

$$\kappa = \rho K(t)$$

$$\frac{d\kappa}{dt} = \rho \frac{dK(t)}{dt}$$

$$\boxed{\frac{d\kappa}{dt} = \rho I(t)} \quad \text{—— (2)}$$



In equilibrium

AD = the potential output

$Y = \kappa$

$$\boxed{\frac{dY}{dt} = \frac{d\kappa}{dt}} \quad \text{—— (3)}$$

I(t)

Find the time path of investment that satisfies this equilibrium condition, $\forall t$.

$$\boxed{k\frac{dI(t)}{dt} = \rho I(t)}$$

$$\frac{1}{I(t)}\frac{dI(t)}{dt} = \rho\frac{1}{k}$$

$$\frac{1}{I(t)}\frac{dI(t)}{dt} = \rho . s \qquad - (4)$$

Integrating both sides w.r.d t

$$\int \frac{1}{I(t)}\frac{dI(t)}{dt}.dt = \int \rho s dt$$

$$\ln|I(t)| + C_1 = \rho s t + C_2$$

$$\ln|I(t)| = \rho s t + C \ ; \ C = C_2 - C_1$$

take exp,

$$e^{\ln|I(t)|} = e^{\rho s t + c}$$

$$I(t) = A . e^{\rho s t} \ ; \ A = e^{c}$$

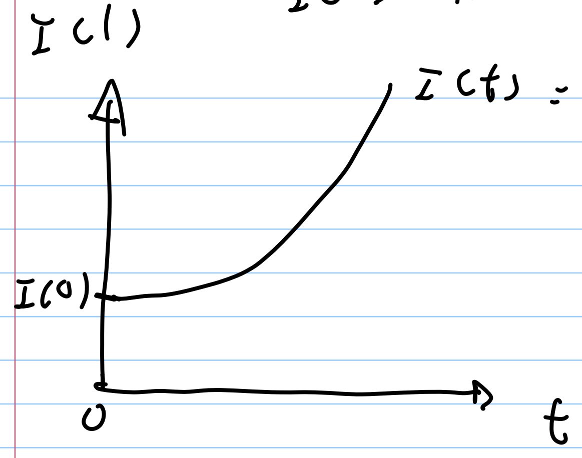
$t = 0$

$$I(0) = A e^{0}$$

$$A = I(0)$$

$$\therefore \boxed{I(t) = I(0) e^{\rho s t}}$$

$I(0)$ is the initial investment



To maintain the balance b/w D and capacity, the flows of investment must grow at the exponential rate of ρs.

ρs = the equilibrium growth rate of $I(t)$

Assuming r = the actual growth rate of $I(t)$

if $r > \rho s$

$$\frac{dY}{dt} = \frac{r}{s} I(0) e^{rt}$$

$$\frac{dK}{dt} = \rho I(t) = \rho I(0) e^{rt}$$

$$\frac{dY/dt}{dK/dt} = \frac{r}{\rho s}$$

$$r > \rho s \Rightarrow \frac{dY}{dt} > \frac{dK}{dt}$$

Demand growth > output growth

⇒ shortage of capacity.

Topic #2 Differential Equations

DE ⇒ (1) The unknown is a f^n, i.e. a f^n of time t.

(2) The equation includes one or more derivatives of the f^n.

Ex

$$\frac{dy(t)}{dt} = a\,y(t) \Rightarrow y(t) = f(t)$$

$$\frac{dy(t)}{dt} + 2t\,y(t) = 4t \Rightarrow \frac{dy}{dt} = -2t\,y(t) + 4t$$

$$\frac{dy}{dt} = y'(t) = \dot{y}(t)$$

$$y'(t) = F(y(t), t)$$

where $y(t)$ is the unknown f^n

The Sol^n : $y(t)$ ⇒ time path of y

2.1 First-order Differential Equations

⇒ only the "first" derivative ⇒ $\frac{dy}{dt}$, $\left[\frac{dy(t)}{dt}\right]^2$

⇒ focus on 'linear'

nonlinear $\Rightarrow$ $y(t)\,y'(t)$

General form

$$y'(t) + u(t).y(t) = w(t) \qquad -(1)$$

$\left.\begin{matrix} u(t) \\ w(t) \end{matrix}\right\}$ are f^{n} of t

and could be constants.

2.1.1 constant coefficent and constant term

$$u(t) = a \quad \text{and} \quad w(t) = b$$

$$y'(t) + a.y(t) = b$$

© **Homogeneous Case**

when $b = 0$

The reduced equation

$$\left[\begin{aligned} y'(t) + ay(t) &= 0 \qquad -(2) \end{aligned}\right.$$

$$y'(t) = -ay(t)$$

$$\frac{1}{y(t)}\frac{dy(t)}{dt} = -a$$

$$\int \frac{1}{y(t)}\frac{dy(t)}{dt}\,dt = \int -a\,dt$$

$$\ln|y| + c_1 = -at + c_2$$

General Sol^n $\boxed{y(t) = A\, e^{-at}}$ — (3)

Definite Sol^n At $t=0$

$$y(0) = A e^{0}$$

$$y(0) = A$$

$$\therefore \quad \boxed{y(t) = y(0)\, e^{-at}} \quad —(3)'$$

ⓔ The Nonhomogeneous Case

$u = a$ and $w = b$

complete equation $\left[\frac{dy(t)}{dt} + a\,y(t) = b\right.$ — (4).

(6)

$$y(t) = y_c + y_p$$

y_c = the complementary f^n

⇒ the sol^n when $b = 0$

$$\therefore \quad y_c = A e^{-at}$$

y_p = the particular integral

To find y_p, we propose that

$$y_p = y(t) = k$$

$$y'(t) = 0$$

sub. into (4) $\quad 0 + a . y_p = b$

$$y_p = \frac{b}{a}$$

Note that ; $b \neq 0$ and $a \neq 0$

As long as $a \neq 0 \quad y_p = k = \frac{b}{a}$

<u>General Soln</u> . $\quad y(t) = y_c + y_p$

$$\boxed{y(t) = Ae^{-at} + \frac{b}{a}} \quad \text{— (5)}$$

$a \neq 0$

<u>Definite Soln</u>

@ $t = 0 \quad y(0) = Ae^{0} + \frac{b}{a}$

$$\therefore A = y(0) - \frac{b}{a}$$

sub into (5)

$$\boxed{y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}} \quad \text{— (6)}$$

$a \neq 0$

Verification of the Sol^n (6)

Diff (6) w.r.t. t

$$y'(t) = (-a)\left[y(0) - \frac{b}{a}\right]e^{-at}$$

sub into LHS of (4); the 1st term

$$\underbrace{(-a)\left[y(0)-\frac{b}{a}\right]e^{-at}}_{y'(t)} + a\underbrace{\left\{\left[y(0)-\frac{b}{a}\right]e^{-at} + \frac{b}{a}\right\}}_{y(t)} = b$$

$$y'(t) + a\, y(t) = b$$

$$\cancel{a}\left(\frac{b}{\cancel{a}}\right) = b$$

$\therefore$ (6) is the sol^n to (4).

Ex $\dfrac{dy}{dt} + 2y - 8 = 0$; $\dfrac{dy}{dt} + ay = b$

$a = 2$

$b = 8$

not -8

$$y(t) = y_c + y_p$$

$$y_c = \left[y(0) - \frac{b}{a}\right]e^{-at}$$

$$= \left[y(0) - \frac{8}{2}\right]e^{-2t}$$

$$y_p = \frac{b}{a} = \frac{8}{2} = 4$$

$$y(t) = (y(0) - 4)e^{-2t} + 4$$

What if $t \to \infty$, $y(t) \to ?$

Eqn (4) again

$$y'(t) + a y(t) = b \qquad - (4)$$

if $a = 0$ $\quad y(t) = ?$

$$y'(t) = b$$

$$\boxed{y(t) = bt + c}$$

$$\boxed{y(t) = y_c + y_p}$$

$y_c = A e^{-at} \Rightarrow a = 0 \Rightarrow y_c = A$

$y_p = \frac{b}{a} \Rightarrow$ not defined

assume $y_p = k$

assume $y_p = kt$

$$y'(t) = k$$

$\therefore$

$$y'(t) + a y(t) = b$$

$$k + 0.(kt) = b$$

$$k = b$$

$$\therefore \quad y_p = bt$$

$$\boxed{y(t) = A + bt}$$

When $a = 0$ and $b \neq 0$

General Sol^n $\quad y(t) = A + bt \qquad -(7)$

Def^n Sol^n $\quad y(t) = y(0) + bt \qquad -(8)$

Ex. $\dfrac{dy}{dt} = 2$ and $y(0)\ 5$

$$\therefore \quad y(t) = 5 + 2t$$

2.1.2. <u>Dynamics of Market Price</u>

A

$$D: \quad Q_d = \alpha' - \beta' P \qquad (\alpha, \beta > 0)$$

$$S: \quad Q = -\gamma' + \delta P \qquad (\gamma', \delta > 0)$$

$$\Bigg\} - (9)$$

$$P^* = \frac{\alpha + \gamma}{\beta + \delta} \qquad -(10)$$

if $P(0) = P^*$ the mkt is in equilibrium

no dynamic adjustment

$P(0) \neq P^*$

$P(0) > P^*$

$P(0) < P^*$