

Solution Part III: Exercise for Assignment 5

1. Let $X = \{2, 3, 4\}$ and $Y = \{1, 5\}$. Define relations

$f : X \rightarrow Y$ by $f = \{(x, y) \in X \times Y | x > y\}$,

$g : X \rightarrow Y$ by $g = \{(x, y) \in X \times Y | x = 2y\}$, and

$h : X \rightarrow Y$ by $h = \{(x, y) \in X \times Y | y = x^2\}$.

- List all the elements of the Cartesian product $X \times Y$.
- List all the elements of the relations f , g , and h .
- Determine which of the relations f, g, h is a function from X to Y . Explain your answer.
- What is the inverse image of 1 under f ?
- What is the inverse image of 5 under f ?

Solution:

- List all the elements of the Cartesian product $X \times Y$.

$$X \times Y = \{(2, 1), (2, 5), (3, 1), (3, 5), (4, 1), (4, 5)\}$$

- List all the elements of the relations f , g , and h .

$$f = \{(2, 1), (3, 1), (4, 1)\}.$$

$$g = \{(2, 1)\}$$

$$h = \{\} = \emptyset$$

- Determine which of the relations f, g, h is a function. Explain your answer.

-The relation f is a function because all the elements in the domain X get mapped to an element in Y and each of them gets mapped only once (even though the element 1 in Y gets mapped more than once).

-The relation g is not a function from X to Y , since there are some element in X doesn't get mapped to Y (i.e. $x = 3, 4$).

-The relation h is not a function from X to Y , since none of the elements in X gets mapped (i.e. $x = 2, 3, 4$) to Y .

- What is the inverse image of 1 under f ?

$$\{x \in X | (x, 1) \in f\} = \{2, 3, 4\}$$

- What is the inverse image of 5 under f ?

$$\{x \in X | (x, 5) \in f\} = \emptyset$$

2. Let $A = \{1, 2, 3\}$ and let $\mathcal{P}(A)$ be the set of all subsets of the set A . Define a relation r and s as

$$r = \{(x, y) \in A \times \mathcal{P}(A) \mid x = \text{the number of elements in } y\},$$

$$s = \{(u, v) \in \mathcal{P}(A) \times A \mid (v, u) \in r\}.$$

- (a) Draw arrow diagrams of r and s .
 (b) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.
 (c) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.

Solution:

- (a) Draw an arrow diagrams of r and s .

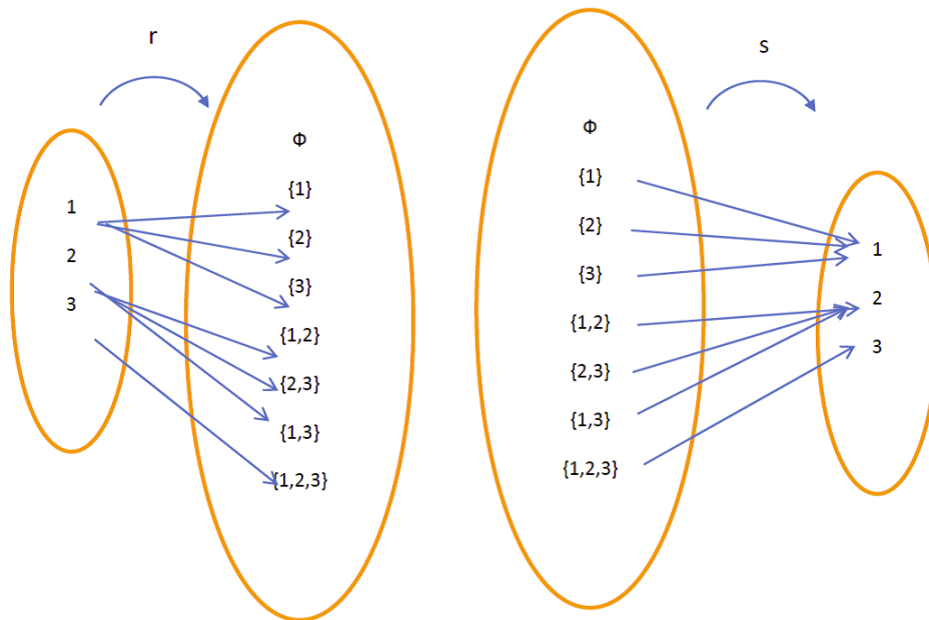


Figure 1: Problem 2(a)

- (b) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.
 No, r is not a function because the element 1 in A gets mapped to $\{1\}, \{2\}, \{3\} \in \mathcal{P}(A)$ (i.e. it gets mapped more than once and similarly for $2 \in A$).
 (c) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.
 No, s is not a function because the element $\emptyset \in \mathcal{P}(A)$ does not get mapped to any element in A .

3. Define $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (y, x - 2) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

- (a) Is H one-to-one? Prove or give a counterexample.

- (b) Is H onto? Prove or give a counterexample.
 (c) Is H bijective? If so, find H^{-1} , the inverse function of H .

Solution:

- (a) Is H one-to-one? Prove or give a counterexample.

Solution: Yes, H is one-to-one. Let $(x_1, y_1), (x_2, y_2)$ be some elements in the domain $\mathbb{R} \times \mathbb{R}$. We want to show that if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$. Suppose $H(x_1, y_1) = H(x_2, y_2)$. Then

$$(y_1, x_1 - 2) = (y_2, x_2 - 2)$$

or equivalently, $y_1 = y_2$ and $x_1 - 2 = x_2 - 2$. I.e.,

$$x_1 - 2 = x_2 - 2 \quad \Rightarrow \quad x_1 = x_2$$

That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$ and therefore, H is one-to-one.

- (b) Is H onto? Prove or give a counterexample.

Solution: Yes, H is not onto. Notice that if we pick (u, v) from the co-domain $\mathbb{R} \times \mathbb{R}$ and suppose that there is (x, y) in the domain such that $H(x, y) = (u, v)$, then

$$(u, v) = H(x, y) = (y, x - 2)$$

or we must have

$$u = y \quad \Rightarrow \quad y = u \in \mathbb{R}$$

and

$$v = x - 2 \quad \Rightarrow \quad x = v + 2 \in \mathbb{R}.$$

That is, for a given (u, v) from the co-domain $\mathbb{R} \times \mathbb{R}$ we can use $(x, y) = (v + 2, u)$ so that

$$H(x, y) = H(v + 2, u) = (u, (v + 2) - 2) = (u, v).$$

Therefore, H is onto.

- (c) Is H bijective? If so, find H^{-1} , the inverse function of H .

Solution: Yes.

From (a) and (b), since H is both one-to-one and onto, then H bijective. To find the inverse function, H^{-1} , of H , we recall from the definition

$$H^{-1}(u, v) = (x, y) \quad \Leftrightarrow \quad H(x, y) = (u, v).$$

Since we have from (b) that

$$H(v + 2, u) = (u, v),$$

and hence the inverse function $H^{-1} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ is given by

$$H^{-1}(u, v) = (v + 2, u).$$

4. Define functions f and g as follows:

$$f = \{(1, 10), (3, 30), (5, 50)\} \text{ and}$$

$$g = \{(10, k), (20, \ell), (30, m), (40, n), (50, t)\}.$$

- (a) Determine the domain and range for each of functions f and g .
- (b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.
- (c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.

Solution:

- (a) Determine the domain and range for each of functions f and g .

$$D_f = \{1, 3, 5\}, \quad R_f = \{10, 30, 50\},$$

$$D_g = \{10, 20, 30, 40, 50\}, \quad R_g = \{k, \ell, m, n, t\}$$

- (b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.

$\underline{g \circ f}$: Since $R_f \cap D_g = \{10, 30, 50\} \neq \emptyset$, we can construct $g \circ f$:

$$(g \circ f)(1) = g(f(1)) = g(10) = k$$

$$(g \circ f)(3) = g(f(3)) = g(30) = m$$

$$(g \circ f)(5) = g(f(5)) = g(50) = t.$$

That is, $g \circ f = \{(10, k), (30, k), (50, t)\}$.

The domain is $\{10, 30, 50\}$ and the range is $\{k, m, t\}$.

$\underline{f \circ g}$: Since $R_g \cap D_f = \{k, \ell, m, n, t\} \cap \{1, 3, 5\} = \emptyset$, we cannot construct $f \circ g$.

- (c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.
First notice that g^{-1} is a function since g is bijective.

$$g^{-1} = \{(k, 10), (\ell, 20), (m, 30), (n, 40), (t, 50)\}.$$

That is, $D_{g^{-1}} = \{k, \ell, m, n, t\}$ and $R_{g^{-1}} = \{10, 20, 30, 40, 50\}$.

$\underline{g \circ g^{-1}}$: Since $R_{g^{-1}} \cap D_g = \{10, 20, 30, 40, 50\} \neq \emptyset$, we can construct $g \circ g^{-1}$:

$$(g \circ g^{-1})(k) = g(g^{-1}(k)) = g(10) = k$$

$$(g \circ g^{-1})(\ell) = g(g^{-1}(\ell)) = g(20) = \ell$$

$$(g \circ g^{-1})(m) = g(g^{-1}(m)) = g(30) = m$$

$$(g \circ g^{-1})(n) = g(g^{-1}(n)) = g(40) = n$$

$$(g \circ g^{-1})(t) = g(g^{-1}(t)) = g(50) = t.$$

That is,

$$g \circ g^{-1} = \{(k, k), (\ell, \ell), (m, m), (n, n), (t, t)\}.$$

That is, $D_{g \circ g^{-1}} = R_{g \circ g^{-1}} = \{k, \ell, m, n, t\}$.

$\underline{g^{-1} \circ g}$: Since $D_{g^{-1}} \cap R_g = \{k, \ell, m, n, t\} \neq \emptyset$, we can construct $g^{-1} \circ g$:

$$(g^{-1} \circ g)(10) = g^{-1}(g(10)) = g^{-1}(k) = 10$$

$$(g^{-1} \circ g)(20) = g^{-1}(g(20)) = g^{-1}(\ell) = 20$$

$$(g^{-1} \circ g)(30) = g^{-1}(g(30)) = g^{-1}(m) = 30$$

$$(g^{-1} \circ g)(40) = g^{-1}(g(40)) = g^{-1}(n) = 40$$

$$(g^{-1} \circ g)(50) = g^{-1}(g(50)) = g^{-1}(t) = 50.$$

That is,

$$g^{-1} \circ g = \{(10, 10), (20, 20), (30, 30), (40, 40), (50, 50)\}.$$

$$\text{That is, } D_{g^{-1} \circ g} = R_{g^{-1} \circ g} = \{10, 20, 30, 40, 50\}.$$

5. Define

$$g(x) = \frac{1}{\sqrt{x+1}} + 1 \quad \text{and} \quad F(x) = \begin{cases} x^2 - 1, & x \in [-3, 1) \\ 2 - x, & x \in [1, 4]. \end{cases}$$

- Find the domain and range for each of the functions g and F .
- Construct the composite functions $F \circ g$, and $g \circ F$ (if possible). Determine the domains for these composite functions.
- Is the function F bijective? If so, find the **inverse function** of F . Justify your answer.

Solution:

- Find the domain, co-domain, and range for each of the functions g and F .

To find the domain of g , notice that we must have $x+1 > 0$ or $x > -1$. That is, $x \in (-1, \infty)$.

To find the range of g ,

$$x + 1 > 0 \Rightarrow \sqrt{x+1} > 0 \Rightarrow \frac{1}{\sqrt{x+1}} > 0 \Rightarrow \frac{1}{\sqrt{x+1}} + 1 > 1 \Rightarrow g(x) > 1$$

That is,

$$D_g = (-1, \infty), \quad \text{and} \quad R_g = (1, \infty).$$

To find the domain of F , notice that $x^2 - 1$ is well-defined for $x \in [-3, 1)$ and $2 - x$ is well-defined for $x \in [1, 4]$.

Hence the domain $D_F = [-3, 4]$.

To find the range of F , we consider 2 cases.

- Suppose $x \in [-3, 1)$.

$$\text{If } x \in [-3, -1], -3 \leq x \leq -1 \Rightarrow 1 \leq x^2 \leq 9 \Rightarrow 0 \leq x^2 - 1 \leq 8 \Rightarrow F(x) \in [0, 8].$$

$$\text{If } x \in [-1, 1), -1 \leq x < 1 \Rightarrow 0 \leq x^2 \leq 1 \Rightarrow -1 \leq x^2 - 1 \leq 0 \Rightarrow F(x) \in [-1, 0].$$

$$\text{That is, } F(x) \in [-1, 0] \cup [0, 8] = [-1, 8] \text{ when } x \in [-3, 1).$$

- Suppose $x \in [1, 4]$. Then

$$1 \leq x \leq 4 \Rightarrow -4 \leq -x \leq -1 \Rightarrow -2 \leq 2 - x \leq 1 \Rightarrow F(x) \in [-2, 1].$$

Hence $F(x) \in [-1, 8] \cup [-2, 1] = [-2, 8]$ and the range of F is $R_F = [-2, 8]$.

That is,

$$D_g = (-1, \infty), \quad \text{and} \quad R_g = (1, \infty).$$

$$D_F = [-3, 4], \quad \text{and} \quad R_F = [-2, 8].$$

- (b) Construct the composite functions $F \circ g$, and $g \circ F$ (if possible). Determine the domains for these composite functions.

$F \circ g$: Notice that $R_g \cap D_F = (1, \infty) \cap [-3, 4] \neq \emptyset$. So, we can construct $F \circ g$:

$$(F \circ g)(x) = F(g(x)) = F\left(\frac{1}{\sqrt{x+1}} + 1\right) = 2 - \left(\frac{1}{\sqrt{x+1}} + 1\right) = 1 - \frac{1}{\sqrt{x+1}}.$$

Above, we have used the fact that $\frac{1}{\sqrt{x+1}} + 1 \in R_g = (1, \infty)$ and $\frac{1}{\sqrt{x+1}} + 1 \geq 1$ which implies that the formula in the second condition of F has to be used. To find $D_{F \circ g}$, first note that $D_{F \circ g} \subset D_g = (-1, \infty)$ and we must also have that $x \in D_{F \circ g}$ must satisfy $g(x) \in D_F \cap R_g$

$$\begin{aligned} \Rightarrow \frac{1}{\sqrt{x+1}} + 1 \in [-3, 4] \cap (1, \infty) &\Rightarrow 1 < \frac{1}{\sqrt{x+1}} + 1 \leq 4 \Rightarrow 0 < \frac{1}{\sqrt{x+1}} \leq 3 \Rightarrow \\ &\sqrt{x+1} \geq \frac{1}{3} \Rightarrow x+1 \geq \frac{1}{9} \Rightarrow x \geq -\frac{8}{9} \end{aligned}$$

and the domain of $D_{F \circ g} = [-\frac{8}{9}, \infty) \cap (-1, \infty) = [-\frac{8}{9}, \infty)$. ■

$g \circ F$: Notice that $R_F \cap D_g = [-2, 8] \cap (-1, \infty) \neq \emptyset$. So, we can construct $g \circ F$: for $x \in D_F = [-3, 4]$,

$$(g \circ F)(x) = g(F(x)).$$

- (i) For $x \in [-3, 1)$,

$$(g \circ F)(x) = g(F(x)) = g(x^2 - 1) = \frac{1}{\sqrt{(x^2 - 1) + 1}} + 1 = \frac{1}{\sqrt{x^2}} + 1 = \frac{1}{|x|} + 1.$$

The value of $x \in [-3, 1)$ must also satisfy $F(x) \in (-1, \infty)$ or $x^2 - 1 > -1 \Rightarrow x^2 > 0$. Hence, $x \in [-3, 1) - \{0\} = [-3, 0) \cup (0, 1)$.

- (ii) For $x \in [1, 4]$,

$$(g \circ F)(x) = g(F(x)) = g(2 - x) = \frac{1}{\sqrt{(2 - x) + 1}} + 1 = \frac{1}{\sqrt{3 - x}} + 1.$$

The value of $x \in [1, 4]$ must also satisfy $F(x) \in (-1, \infty)$ or $2 - x > -1 \Rightarrow x < 3$. Hence, $x \in [1, 4] \cap (-\infty, 3) = [1, 3)$.

That is, from (i) and (ii)

$$(g \circ F)(x) = \begin{cases} \frac{1}{|x|} + 1, & x \in [-3, 0) \cup (0, 1) \\ \frac{1}{\sqrt{3-x}} + 1, & x \in [1, 3). \end{cases}$$

and the domain is $D_{g \circ F} = [-3, 0) \cup (0, 1) \cup [1, 3) = [-3, 0) \cup (0, 3)$. ■

- (c) Is the function F bijective? If so, find the **inverse function** of F . Justify your answer.

No. Since F is not injective. E.g. when $x = \pm \frac{1}{\sqrt{2}} \in [-3, 1) \in D_F$,

$$F(-\frac{1}{\sqrt{2}}) = F(\frac{1}{\sqrt{2}}) = \frac{1}{2} - 1 = -\frac{1}{2} \text{ but } -\frac{1}{\sqrt{2}} \neq \frac{1}{\sqrt{2}}.$$