

## Outline

1. Utility Maximization
2. Utility maximization – Tricky Cases
3. Indirect Utility Function
4. Comparative Statics (Introduction)
5. Income Changes

# 1 Utility maximization

- Nicholson, Ch. 4, pp. 119–128
- $X = R_+^2$  (2 goods)
- Consumers: choose bundle  $x = (x_1, x_2)$  in  $X$  which yields highest utility.
- Constraint: income =  $M$
- Price of good 1 =  $p_1$ , price of good 2 =  $p_2$
- Bundle  $x$  is feasible if  $p_1x_1 + p_2x_2 \leq M$
- Consumer maximizes

$$\begin{aligned} & \max_{x_1, x_2} u(x_1, x_2) \\ & s.t. \quad p_1x_1 + p_2x_2 \leq M \\ & \quad \quad x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

- Maximization subject to inequality. How do we solve that?
- Trick:  $u$  strictly increasing in at least one dimension. ( $\succeq$  strictly monotonic)
- Budget constraint always satisfied with equality
- Ignore temporarily  $x_1 \geq 0, x_2 \geq 0$  and check afterwards that they are satisfied for  $x_1^*$  and  $x_2^*$ .

- Problem becomes

$$\begin{aligned} & \max_{x_1, x_2} u(x_1, x_2) \\ & s.t. \quad p_1 x_1 + p_2 x_2 - M = 0 \end{aligned}$$

- $L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - M)$

- F.o.c.s:

$$\begin{aligned} u'_{x_i} - \lambda p_i &= 0 \text{ for } i = 1, 2 \\ p_1 x_1 + p_2 x_2 - M &= 0 \end{aligned}$$

- Moving the two terms across and dividing, we get:

$$MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}$$

- Graphical interpretation.

- Second order conditions:

$$H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix}$$

$$\begin{aligned} |H| &= p_1 \left( -p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1} \right) \\ &\quad - p_2 \left( -p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1} \right) \\ &= -p_1^2 u''_{x_2,x_2} + 2p_1 p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \end{aligned}$$

- Notice:  $u''_{x_2,x_2} < 0$  and  $u''_{x_1,x_1} < 0$  usually satisfied (but check it!).
- Condition  $u''_{x_1,x_2} > 0$  is then sufficient

Example with CES utility function.

$$\begin{aligned} \max_{x_1, x_2} & \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \\ \text{s.t.} & p_1 x_1 + p_2 x_2 - M = 0 \end{aligned}$$

- Lagrangean =

- F.o.c.:

- Solution:

$$x_1^* = \frac{M}{p_1 \left( 1 + \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\rho-1}} \left( \frac{p_2}{p_1} \right)^{\frac{\rho}{\rho-1}} \right)}$$

$$x_2^* = \frac{M}{p_2 \left( 1 + \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\rho-1}} \left( \frac{p_1}{p_2} \right)^{\frac{\rho}{\rho-1}} \right)}$$

- Special case 1:  $\rho = 0$  (Cobb-Douglas)

$$x_1^* = \frac{\alpha}{\alpha + \beta} \frac{M}{p_1}$$

$$x_2^* = \frac{\beta}{\alpha + \beta} \frac{M}{p_2}$$

- Special case 1:  $\rho \rightarrow 1^-$  (Perfect Substitutes)

$$x_1^* = \begin{cases} 0 & \text{if } p_1/p_2 \geq \alpha/\beta \\ M/p_1 & \text{if } p_1/p_2 < \alpha/\beta \end{cases}$$

$$x_2^* = \begin{cases} M/p_2 & \text{if } p_1/p_2 \geq \alpha/\beta \\ 0 & \text{if } p_1/p_2 < \alpha/\beta \end{cases}$$

- Special case 1:  $\rho \rightarrow -\infty$  (Perfect Complements)

$$x_1^* = \frac{M}{p_1 + p_2} = x_2^*$$

- Parameter  $\rho$  indicates substitution pattern between goods:
  - $\rho > 0 \rightarrow$  Goods are (net) substitutes
  - $\rho < 0 \rightarrow$  Goods are (net) complements

## **2 Utility maximization – tricky cases**

1. Non-convex preferences. Example:

2. Example with CES utility function.

$$\begin{aligned} \max_{x_1, x_2} & (\alpha x_1^\rho + \beta x_2^\rho)^{1/\rho} \\ \text{s.t.} & p_1 x_1 + p_2 x_2 - M = 0 \end{aligned}$$

- With  $\rho > 1$  the interior solution is a minimum!
- Draw indifference curves for  $\rho = 1$  (boundary case) and  $\rho = 2$
- Can also check using second order conditions

2. Solution does not satisfy  $x_1^* > 0$  or  $x_2^* > 0$ . Example:

$$\begin{aligned} \max x_1 * (x_2 + 5) \\ \text{s.t. } p_1 x_1 + p_2 x_2 = M \end{aligned}$$

- In this case consider corner conditions: what happens for  $x_1^* = 0$ ? And  $x_2^* = 0$ ?

### 3. Multiplicity of solutions.

- Example 1: Perfect Substitutes with  $p_1/p_2 = \alpha/\beta$
- Example 2: Non-convex preferences with two optima

### 3 Indirect utility function

- Nicholson, Ch. 4, pp. 128-130
- Define the indirect utility  $v(\mathbf{p}, M) \equiv u(\mathbf{x}^*(\mathbf{p}, M))$ , with  $\mathbf{p}$  vector of prices and  $\mathbf{x}^*$  vector of optimal solutions.
- $v(\mathbf{p}, M)$  is the utility at the optimum for prices  $\mathbf{p}$  and income  $M$
- Some comparative statics:  $\partial v(\mathbf{p}, M)/\partial M = ?$
- Hint: Use Envelope Theorem on Lagrangean function

- What is the sign of  $\lambda$ ?
- $\lambda = u'_{x_i}/p > 0$
- $\partial v(\mathbf{p}, M)/\partial p_i = ?$
- Properties:
  - Indirect utility is always increasing in income  $M$
  - Indirect utility is always decreasing in the price  $p_i$

## 4 Comparative Statics (introduction)

- Nicholson, Ch. 5, pp. 145-155
- Utility maximization yields  $x_i^* = x_i^*(p_1, p_2, M)$
- Quantity consumed as a function of income and price
  
- What happens to quantity consumed  $x_i^*$  as prices or income varies?

- Simple case: Equal increase in prices and income.

- $M' = tM, p'_1 = tp_1, p'_2 = tp_2.$

- Compare  $x^*(tM, tp_1, tp_2)$  and  $x^*(M, p_1, p_2).$

- What happens?

- Write budget line:  $tp_1x_1 + tp_2x_2 = tM$

- Demand is homogeneous of degree 0 in  $\mathbf{p}$  and  $M$ :

$$x^*(tM, tp_1, tp_2) = t^0 x^*(M, p_1, p_2) = x^*(M, p_1, p_2).$$

- Consider Cobb-Douglas Case:

$$x_1^* = \frac{\alpha}{\alpha + \beta} M/p_1, x_2^* = \frac{\beta}{\alpha + \beta} M/p_2$$

- What is  $\partial x_1^*/\partial M$ ?

- What is  $\partial x_1^*/\partial p_1$ ?

- What is  $\partial x_1^*/\partial p_2$ ?

- General results?

## 5 Income changes

- Income increases from  $M$  to  $M' > M$ .
- Budget line ( $p_1x_1 + p_2x_2 = M$ ) shifts out:

$$x_2 = \frac{M'}{p_2} - x_1 \frac{p_1}{p_2}$$

- New optimum?

- Engel curve:  $x_i^*(M)$  : demand for good  $i$  as function of income  $M$  holding fixed prices  $p_1, p_2$

- Does  $x_i^*$  increase with  $M$ ?

- Yes. Good  $i$  is *normal*

- No. Good  $i$  is *inferior*